MATH 20A – OVERVIEW – 2D INTEGRAL THEOREMS

Line integral along a curve C of a vector field which is the gradient of a function f

$$\int_C (\nabla f) \cdot d\mathbf{r}$$

• The curve C can be *any curve* but the integrand can't just be anything—must be a *gradient.*

$\longleftrightarrow$ (fund. thm. of line integrals)

$\xrightarrow{\text{more useful direction}}$

Difference of function values at the two endpoints

$$f(\text{end}) - f(\text{start})$$

Region integral of a function which is the curl of a vector function **F** *over region R.*

$$\iint_R (\operatorname{curl} \mathbf{F})\, dx\, dy$$

• The region R can be *any region* but the integrand can't just be anything—must be the *curl* of something.

$\longleftrightarrow$ (Green's thm.—curl form)

$\xleftarrow{\text{more useful direction}}$

Line integral counterclockwise around edge of region

$$\int_C \mathbf{F} \cdot d\mathbf{r} \qquad C = \text{bndry}(R)$$

• The vector field **F** can be *any vector field* but the curve C you integrate over has to be special—it must be the *boundary* of some region. (E.g. it must be a closed curve: no beginning or end.)

Region integral of a function which is the divergence of a vector function **F** *over region R.*

$$\iint_R (\operatorname{div} \mathbf{F})\, dx\, dy$$

• The region R can be *any region* but the integrand can't just be anything—must be the *divergence* of something.

$\longleftrightarrow$ (Green's thm.—divergence form)

$\xleftarrow{\text{more useful direction}}$

Flux integral measuring net flux out of region

$$\int_C \mathbf{F} \cdot R d\mathbf{r} \qquad C = \text{bndry}(R)$$

• The vector field **F** can be *any vector field* but the curve C you integrate over has to be special—it must be the *boundary* of some region. (E.g. it must be a closed curve: no beginning or end.)

Usage guide (if you're doing a line integral)

- Check to see whether the *vector field you're integrating* is the gradient of a function. If it is, use the fundamental theorem of line integrals.
- Check to see whether the *curve you're integrating along* is the boundary of a region. If it is, use Green's theorem (curl form).
- Otherwise, do it directly.

Note that there are some more cunning uses not covered by this simple guide!

Usage guide (if you're doing a flux integral)

- Check to see whether the *curve you're measuring the flux across* is the boundary of a region. If it is, use Green's theorem (divergence form).
- Otherwise, do it directly.

Note that there are some more cunning uses not covered by this simple guide!

MATH 20A – OVERVIEW – 3D INTEGRAL THEOREMS

Line integral along a curve C of a vector field which is the gradient of a function f $$\int_C (\nabla f)\cdot d\mathbf{r}$$ • The curve C can be *any curve* but the integrand can't just be anything—must be a *gradient.*	$\longleftrightarrow$ (fund. thm. of line integrals) $\xrightarrow{\text{more useful direction}}$	*Difference of function values at the two endpoints* $$f(\text{end}) - f(\text{start})$$
Surface flux integral measuring the flux (through a surface S) of a vector field curl $\mathbf{F}$ *which is the curl of a another vector field* $\mathbf{F}$. $$\iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S}$$ • The surface S can be *any surface* but the integrand can't just be anything—must be the *curl* of something.	$\longleftrightarrow$ (Stokes's thm) $\xleftarrow{\text{both directions useful}}$ $\xrightarrow{\text{both directions useful}}$	*Line integral around edge of surface, in direction given by right hand rule* $$\int_C \mathbf{F}\cdot d\mathbf{r} \qquad C = \text{bndry}(S)$$ • The vector field $\mathbf{F}$ can be *any vector field* but the curve C you integrate over has to be special—it must be the *boundary* of some surface. (E.g. it must be a closed curve: no beginning or end.)
Volume integral of a function which is the divergence of a vector function $\mathbf{F}$ *over volume* V. $$\iiint_V (\operatorname{div}\mathbf{F})\,dx\,dy\,dz$$ • The volume V can be *any volume* but the function you integrate can't just be anything—must be the *divergence* of something.	$\longleftrightarrow$ (Divergence thm./ Gauss's theorem) $\xleftarrow{\text{more useful direction}}$	*Surface flux integral measuring net flux out of volume* $$\iint_S \mathbf{F}\cdot d\mathbf{S} \qquad S = \text{bndry}(V)$$ • The vector field $\mathbf{F}$ can be *any vector field* but the surface S you integrate over has to be special—it must be the *boundary* of some 3D region.

Usage guide. (Note that there are some more cunning uses not covered by this simple guide!)

- If you are trying to do a line integral:
  - Check to see whether the *vector field you're integrating* is the gradient of a function. If it is, use the fundamental theorem of line integrals.
  - Check to see whether the *curve you're integrating along* has a begining and an end. If it does, skip onto the next bullet point. If it doesn't, see whether it can be closed off into a region easily. If it can, use Stokes's theorem in the right-to-left direction, especially if the curl of the vector field you have to integrate is much nicer than the original vector field. (If it's not easy to close the curve off the region into a surface, or if the curl is not much nicer than the original vector field, you may be better off doing the integral directly.)
  - Otherwise, do it directly.
- If you are trying to do a surface integral:
  - Check to see whether the *surface you're integrating over* is a boundary of a 3D region. If it is, use the divergence theorem.
  - Check to see whether the *vector field you're integrating* is the curl of a function. If it is, use Stokes' theorem in the left-to-right direction. (If it's not obvious whether the vector field is a curl, guess and check! But if you get bored, go on and do the integral directly.)
  - Otherwise, do it directly.