MATH 20A — FINAL PRACTICE HINTS

For some of the more interesting problems here, I have provided some rather cryptic hints. These may be helpful for you I have also provided solutions (on the next page) to the first 9 problems, and numerical answers for the rest.

4 Use the fact that if θ, ϕ are angles less than π but more than 0, and $\cot\theta = \cot\phi$, then $\theta = \phi$.

5 Use the fundamental theorem of line integrals.

7 First use the divergence theorem

10 You are NOT meant to integrate $\cos(y^2)$. You need to do something else.

11 Use the divergence theorem in an unexpected direction.

MATH 20A — FINAL PRACTICE ANSWERS

1. *Via divergence theorem.* The divergence is 1. Integrating 1 over the sphere given will just give us the volume of the sphere; this sphere has radius 3, so the volume is $(4/3)\pi r^3 = (4/3)\pi\, 3^3 = 36\pi$

   Directly We can parameterize the sphere of radius 3 as:

$$\mathbf{r}(u,v) = \langle 3\sin u\cos v, 3\sin u\sin v, 3\cos u\rangle$$

   where $0 \le u \le \pi, 0 \le v \le 2\pi$. Then we have:

$$\begin{aligned}\mathbf{r}_u &= \langle 3\cos u\cos v, 3\cos u\sin v, -3\sin u\rangle\\ \mathbf{r}_v &= \langle -3\sin u\sin v, 3\sin u\cos v, 3\cos u\rangle\\ \mathbf{r}_u\times\mathbf{r}_v &= \langle 9(\sin u)^2\cos v, 9(\sin u)^2\sin v, 9\sin u\cos u\rangle\end{aligned}$$

   and we can see by drawing a picture that this is the outward normal. Then, writing $\mathbf{F}$ for the vector field $\langle 0,0,x+z\rangle$,

$$\mathbf{F}(\mathbf{r}(u,v)) = \langle 0,0,3\sin u\cos v + 3\cos u\rangle$$

   So

$$\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v) = 27(\sin u)^2\cos u\cos v + 27\sin u(\cos u)^2$$

   and

$$\begin{aligned}\text{answer} &= \int_0^\pi\int_0^{2\pi}\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)dv\,du\\ &= \int_0^\pi\int_0^{2\pi} 27(\sin u)^2\cos u\cos v + 27\sin u(\cos u)^2 dv\,du\\ &= \int_0^\pi 0 + (2\pi)(27)\sin u(\cos u)^2 du\\ &= \int_1^{-1}(2\pi)(27) - t^2 dt\\ &= \int_{-1}^1 (2\pi)(27)t^2 dt\\ &= (2\pi)(27)\frac{1}{3}(1^3-(-1)^3)\\ &= 36\pi\end{aligned}$$

2. We calculate

$$f_x = 3y + 2xy + y^2, \quad f_y = 3x + 2xy + x^2$$

   We must set these equal to zero and solve.

   We start with the first equation, which factorizes as $y(3+2x+y) = 0$; we must consider separately two possibilities.

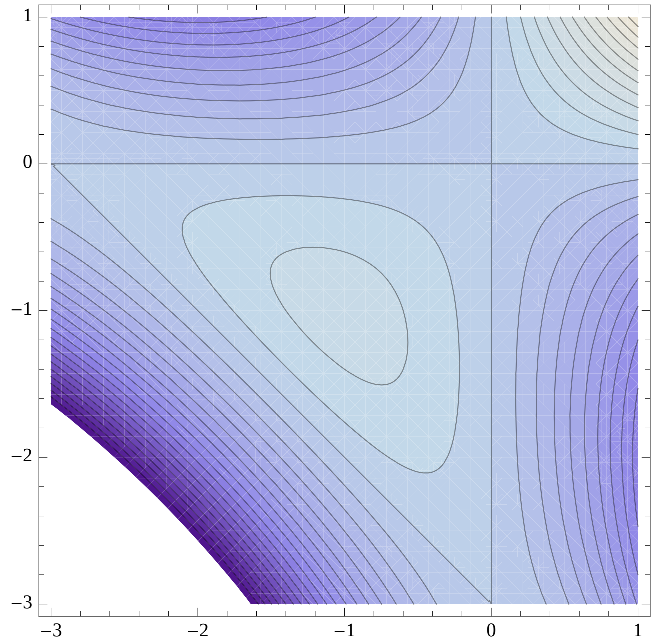
   - $y = 0$. We now turn to the second equation, $3x + 2xy + x^2 = 0$, which becomes $3x + x^2 = 0$. We solve this and get that $x = 0$ or $x = -3$. Thus points are (-3,0), (0,0).
   - $3 + 2x + y = 0$. We now turn to the second equation, $3x + 2xy + x^2 = 0$, which factorizes as $x(3+2y+x) = 0$. We consider separately two sub-possibilities:
     - $x = 0$. Since we know $3 + 2x + y = 0$, we get that $y = -3$, giving the point $(0,-3)$.
     - $3 + 2y + x = 0$. Using this, together with $3 + 2x + y = 0$ (which we also know), gives $x = y = -1$, and we get the point (-1,-1).

   We work out $f_{xx}f_{yy} - (f_{xy})^2 = (2y)(2x) - (3+2x+2y)^2$ at each of these points

point	(0,0)	(-3,0)	(0,-3)	(-1,-1)
$f_{xx}f_{yy} - (f_{xy})^2$	-9	-9	-9	3

This immediately tells us (0,0), (-3,0) and (0,-3) are saddles, while (-1,-1) is either a min or a max. Finally, we calculate that f_{xx} is -2 (which is negative) at (-1,-1), so it must be a max.

Here is a sketch:



3. We calculate

$$f_x = 4 + 2x, \quad f_y = 3y^2 - 6y$$

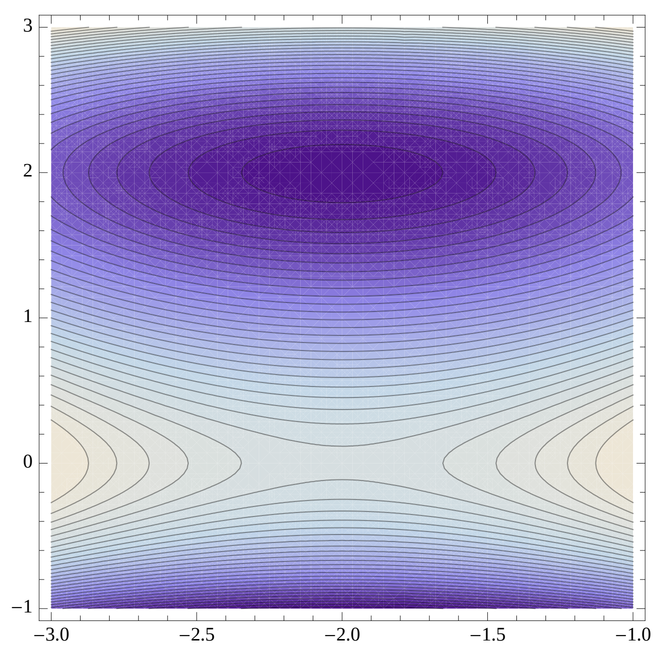
We must set these equal to zero and solve. The first tells us immediately that $x = -2$ at any critical point. The second factorizes, giving us $3y(y-2)$, so $y = 0$ or $y = 2$. Thus the critical points are $(-2, 2)$ or $(-2, 0)$.

We work out $f_{xx} = 2$ and $f_{xx}f_{yy} - (f_{xy})^2 = (2)(6y - 6)$ at each of these points, getting

point	(-2,2)	(-2,0)
$f_{xx}f_{yy} - (f_{xy})^2$	12	-12
f_{xx}	2	2

This tells us that (-2,2) is a minimum and (-2,0) is a saddle.

Here is a sketch:



4. We use Lagrange multipliers; we maximize $f(\alpha, \beta, \gamma)$ subject to $g(\alpha, \beta, \gamma) = 0$, where we define $g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - \pi$. Thus we introduce

$$\begin{aligned} L(\alpha, \beta, \gamma, \lambda) &= f(\alpha, \beta, \gamma) - \lambda g(\alpha, \beta, \gamma) \\ &= \log(\sin(\alpha)\sin(\beta)\sin(\gamma)) - \lambda(\alpha + \beta + \gamma - \pi) \end{aligned}$$

and find the derivative:

$$\partial L/\partial \alpha = \frac{\cos(\alpha)\sin(\beta)\sin(\gamma)}{\sin(\alpha)\sin(\beta)\sin(\gamma)} - \lambda = \cot(\alpha) - \lambda$$

and set this to 0, getting $\lambda = \cot(\alpha)$. SImilarly, $\lambda = \cot(\beta)$ and $\lambda = \cot(\gamma)$, so that $\cot(\alpha) = \cot(\beta) = \cot(\gamma)$.

Now, we know that the angles α, β, γ are between 0 and π, and we can draw a graph of the cotangent function and see if the cotangents of two angles in this range are the same, then the angles themselves must be. Thus $\alpha = \beta = \gamma$, and since they must add to π, we deduce each angle is equal to $\pi/3$.

5. It is difficult to do the line integral directly. A better approach is to observe that the vector field $\mathbf{F}$ is conservative, and indeed $\mathbf{F}$ may be written as ∇f, where:

$$f = xe^{y^2} + \sin(xz)$$

Thus the fine integral of $\mathbf{F}$ along any path can be computed by taking the difference of the values of f at the beginning and end of the path. The path we are given in the question starts at $\langle 0, 1, 0 \rangle$ and ends at $\langle \pi/2, 0, 1 \rangle$ (just plug in $t = 0$ and $t = 1$). Then the answer is

$$f(\text{end}) - f(\text{start}) = (\pi/2 + \sin(\pi/2)) - 0 = \pi/2 + 1$$

6. Let's try to calculate the point P in terms of its coordinates (x, y). The quantity we are trying to minimize then becomes $x^2 + (y-2)^2 + 2(x^2 + y^2)$, which is

$$3x^2 + 3y^2 - 4y + 4$$

which we can call $f(x, y)$. We're trying to minimize this subject to the constraint that $x = y - 1$. If we introduce the function $g(x, y) = 1 + x - y$, then our constraint can be expressed by saying $g = 0$.

We can now use Lagrange multipliers. We introduce

$$\begin{aligned} L(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= 3x^2 + 3y^2 - 4y + 4 + \lambda(1 + x - y) \end{aligned}$$

and find the derivatives and set them equal to 0:

$$\begin{aligned} &x \text{ deriv} & 6x + \lambda &= 0 & (1) \\ &y \text{ deriv} & 6y - 4 - \lambda &= 0 & (2) \\ &\lambda \text{ deriv} & 1 + x - y &= 0 & (3) \end{aligned}$$

Now we must solve these equations. Taking eq (1) minus eq (2) gives

$$6x - 6y + 4 + 2\lambda = 0$$

and then subtracting 6 times equation (3) gives:

$$-6 + 4 + 2\lambda = 0$$

which tells us $\lambda = 1$. Then equation (1) tells us $x = -1/6$ and eq (2) tells us $y = 5/6$. Thus the best possible P is $(-1/6, 5/6)$.

7. By applying the divergence theorem, we know that the flux integral involved in the rules of the 'game' will be the same as the integral of the divergence of $\mathbf{F}$ over the solid $0 \leq x \leq a, 0 \leq y \leq b, , 0 \leq z \leq 1$.

   We can calculate the divergence as $-2x - 4y - z + 12$; so the integral of the divergence over the region will be

$$\begin{aligned}\int_0^a \int_0^b \int_0^1 -2x - 4y - z + 12 dz\, dy\, dx &= \int_0^a \int_0^b -2x - 4y - (1/2) + 12 dy\, dx \\ &= \int_0^a -2xb - 2b^2 + \frac{23b}{2} dx \\ &= -a^2 b - 2ab^2 + \frac{23ab}{2}\end{aligned}$$

   We are trying to maximize this quantity, which is a function of a and b. Let's call this function $f(a, b)$. We can start by finding the critical points... Setting the derivative $\partial f/\partial a$ equal to zero gives

$$0 = -2ab - 2b^2 + \frac{23b}{2} = b(-2a - 2b + (23/2))$$

   We then have two cases:

   - $b = 0$. Setting the other derivative equal to zero gives:

$$0 = -a^2 - 4ab + \frac{23b}{2} = a(-a - 4b + (23/2))$$

     Which gives us two sub-cases:

     - $a = 0$. This gives the point (0,0)
     - $-a - 4b + (23/2) = 0$. This gives the point (23/2,0)

   - $-2a - 2b + (23/2) = 0$. Again the other derivative being zero gives $0 = a(-a - 4b + (23/2))$, and we have two sub-cases:

     - $a = 0$. This gives the point (0,23/4).
     - $-a-4b+(23/2) = 0$. This tells us that $a = -4b+(23/2)$; plugging this into $-2a-2b+(23/2) = 0$, we get $-2(-4b+(23/2))-2b+(23/2) = 0$, so $8b-23-2b+23/2 = 0$, so $6b = 23/2$, so $b = 23/12$. Then $a = 23/6$. This gives the point (23/6,23/12).

   We see that f is in fact zero at all these points except $(23/6, 23/12)$, where it is positive. Moreover, f will be zero at any point on the boundary (where $a = 0$ or $b = 0$). Thus $a = 23/6, b = 23/12$ is the best possible value.[1]

8. (a) a is the height of the cone; b is the radius of the base.

   (b) Several things will work; one is:

$$\mathbf{r}(u, v) = \langle b(1-u)\sin v, b(1-u)\cos v, au\rangle$$

   for $0 \leq u \leq 1, 0 \leq v \leq 2\pi$

   (c) We work out

$$\begin{aligned}\mathbf{r}_u &= \langle -b\sin v, -b\cos v, a\rangle \\ \mathbf{r}_v &= \langle b(1-u)\cos v, -b(1-u)\sin v, 0\rangle \\ \mathbf{r}_u \times \mathbf{r}_v &= \langle ab(\sin v)(1-u), ab(\cos v)(1-u), b^2(1-u)\rangle \\ |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{a^2b^2(1-u)^2(\sin^2 v + \cos^2 v) + b^4(1-u)^2} \\ &= b(1-u)\sqrt{a^2+b^2}\end{aligned}$$

[1] Technically, we should worry about the fact that the range of possible a and b values isn't closed and bounded—I should have written something in the question to say that you were allowed to assume that the critical point with the largest f value was a maximum.

Then the area of the slanty section is:

$$\int_0^1 \int_0^{2\pi} b(1-u)\sqrt{a^2+b^2}\, dv\, du = 2\pi \int_0^1 b(1-u)\sqrt{a^2+b^2}\, du$$
$$= \pi b\sqrt{a^2+b^2}$$

And to get the surface area of the whole cone, we add in the πb^2 surface area of the bottom circle; this gives

$$\text{surface area} = \pi b\sqrt{a^2+b^2} + \pi b^2$$

(d) We are trying to find the volume between the graph of the function $f(x,y) = a(1-\sqrt{\frac{x^2+y^2}{b^2}})$ and the plane $z=0$. We can do this by integrating the function f over the circle $r \leq b$. We get the following integral in polar coordinates

$$\int_0^b \int_0^{2\pi} a(1-\frac{r}{b})r\, d\theta\, dr = 2\pi \int_0^b a(1-\frac{r}{b})r\, dr$$
$$= 2\pi \left[\frac{ar^2}{2} - \frac{ar^3}{3b}\right]_0^b = 2\pi\frac{ab^2}{2} - \frac{ab^2}{3}$$
$$= \pi ab^2/3$$

(e) We are to maximize $f(a,b) = \pi ab^2/3$, subject to the constraint that $g(a,b)=0$, where we define the function g by:

$$g(a,b) = b\pi\sqrt{a^2+b^2} + \pi b^2 - \pi$$

(so g being 0 ensures the surface area is π). We use Lagrange. Introduce:

$$L(a,b,\lambda) = \pi ab^2/3 - \lambda(b\pi\sqrt{a^2+b^2} + \pi b^2 - \pi)$$

we must set all the partials equal to 0. We get the rather horrible looking:

$$0 = \frac{b^2}{3} - \lambda\frac{ab}{\sqrt{a^2+b^2}} \tag{1}$$
$$0 = \frac{2ab}{3} - \lambda\left(\frac{a^2+2b^2}{\sqrt{a^2+b^2}} + 2b\right) \tag{2}$$
$$1 = b\sqrt{a^2+b^2} + b^2 \tag{3}$$

(I've divided these equations out by π to simplify them a bit.)
The first equation tells us that $\lambda = \frac{b^2\sqrt{a^2+b^2}}{3ab}$ (it is ok to divide through by a and b because the question specifically says that they are positive, so they cannot be zero). Plugging this into the second equation gives us:

$$\frac{2ab}{3} - \frac{b^2\sqrt{a^2+b^2}}{3ab}\left(\frac{a^2+2b^2}{\sqrt{a^2+b^2}} + 2b\right) = 0$$

so

$$\frac{2ab}{3} - \frac{b^2}{3ab}(a^2+2b^2+2b\sqrt{a^2+b^2}) = 0$$

so

$$2a^2b^2 - b^2(a^2+2b^2+2b\sqrt{a^2+b^2}) = 0$$

so, since $b \neq 0$ (again, because the question says b must be positive)

$$2a^2 - (a^2+2b^2+2b\sqrt{a^2+b^2}) = 0$$

which simplifies to

$$a^2 = 2b^2 + 2b\sqrt{a^2+b^2} \quad (4)$$

Now, way up above we have equation (3), which we haven't used yet. It tells us that $b\sqrt{a^2+b^2} = (1-b^2)$, which lets us simplify equation (4) to

$$a^2 = 2b^2 + 2(1-b^2)$$

which multiplies out to give $a^2 = 2$, so $a = \sqrt{2}$. Then plugging this into eq (3) gives us

$$1 = b\sqrt{2 + b^2} + b^2$$

which gives $1 - b^2 = b\sqrt{2 + b^2}$; after squaring both sides, we get

$$1 - 2b^2 + b^4 = b^2(2 + b^2)$$

which simplifies to give $4b^2 = 1$, or $b = 1/2$. Thus the critical point is $(a, b) = (1/\sqrt{2}, 1/2)$. We are told that we can assume this is the maximum with no further work, so we are done.

9. We first find the curl. It is

$$\begin{aligned} f = \frac{\partial \mathbf{F}_y}{\partial x} - \frac{\partial \mathbf{F}_x}{\partial y} &= (\frac{x^3}{3} - \frac{x^2}{2}) - (-2y^3 + \frac{y^2}{2}) \\ &= \frac{x^3}{3} - \frac{x^2}{2} + 2y^3 - \frac{y^2}{2} \end{aligned}$$

Let us go through the IFR steps:

- *Is the region closed and bounded?* Yes, it is.
- *Find all boundary pieces, and corners.* The only boundary piece is the circle where $x^2 + y^2 = 4$.
- *Set $\nabla f = 0$ to find critical points in the interior.* The partials are

$$\frac{\partial f}{\partial x} = x^2 - x, \quad \frac{\partial f}{\partial y} = 6y^2 - y$$

  Thus we must solve $x^2 - x = 0$, and $6y^2 - y = 0$. Let's start with $x^2 - x = 0$. Factorizing, we get $(x - 1)x = 0$, so $x = 0$ or $x = 1$.
  - If $x = 0$, then $6y^2 - y = 0$ tells us $(6y - 1)y = 0$, so $y = 0$ or $y = 1/6$. Thus the points are $(0, 0)$ and $(0, 1/6)$.
  - If $x = 1$, then solving for y is in fact exactly the same as it was before, and we get $y = 0$ or $y = 1/6$. Thus the points are $(1, 0)$ and $(1, 1/6)$.

  We can see that all these points lie inside the circle $x^2 + y^2 = 4$, so we must include them all. We work out the function f at these points:

point	(0,0)	(0,1/6)	(1,0)	(1,1/6)
f	0	$-\frac{1}{216}$	$-\frac{1}{6}$	$-\frac{37}{216}$

- *Find critical points on the boundary.* We must use Lagrange multipliers. We introduce

$$L(x, y, \lambda) = \frac{x^3}{3} - \frac{x^2}{2} + 2y^3 - \frac{y^2}{2} + \lambda(x^2 + y^2 - 4)$$

  and set all the partials equal to zero. We get

$$\begin{aligned} x^2 - x + 2\lambda x &= 0 \quad (1) \\ 6y^2 - y + 2\lambda y &= 0 \quad (2) \\ x^2 + y^2 &= 4 \quad (3) \end{aligned}$$

  Let's start solving with the first equation, which factorizes as $x(x - 1 + 2\lambda) = 0$. Thus we have two cases:
  - $x = 0$. Then equation (3) tells us $y = 2$ or $y = -2$. This gives us points $(0, 2)$ and $(0, -2)$.
  - $1 - x = 2\lambda$. We now use equation (2), which factorizes as $y(6y - 1 + 2\lambda) = 0$. Thus we have two sub-cases:
    * $y = 0$. Then equation (3) tells us $x = 2$ or $x = -2$. This gives us points $(2, 0)$ and $(-2, 0)$.

  * $6y - 1 + 2\lambda = 0$. Combining this with $1 - x = 2\lambda$, we get

$$6y - 1 + 1 - x = 0$$

so $x = 6y$. Then equation (3) tells us that $37y^2 = 4$, so $y = \pm 2/\sqrt{37}$; this give points $(12/\sqrt{37}, 2/\sqrt{37})$ and $(-12/\sqrt{37}, -2/\sqrt{37})$.

We work out the function f at these points:

point	(0,2)	(0,-2)	(2,0)	(-2,0)	$(12/\sqrt{37}, 2/\sqrt{37})$	$(-12/\sqrt{37}, -2/\sqrt{37})$
f	14	-18	2/3	-14/3	0.630384	-4.63038

- *Work out function at corners.* There are no corners.
- *Whichever the biggest value seen is, is the maximum.* The largest value was 14, seen at (0,2).

10. The trick is to visualize the region being integrated over, and switch the order of integration, so that you get an order which can be done. Drawing a picture of the volume we're integrating over, then changing to do x outermost, then y, then z, we get that the answer is

$$\begin{aligned}
\int_1^3 \int_0^4 \int_0^{\sqrt{y}} xz\cos(y^2)dz\,dy\,dx &= \int_1^3 \int_0^4 x\frac{z^2}{2}\cos(y^2)|_0^{\sqrt{y}}\,dy\,dx \\
&= \int_1^3 \int_0^4 \frac{1}{2}xy\cos(y^2)\,dy\,dx \\
&= \int_1^3 \frac{1}{4}x\sin(y^2)|_{y=0}^4\,dx \\
&= \int_1^3 \frac{1}{4}x\sin(16)\,dx \\
&= \frac{1}{8}(3^2 - 1^2)\sin(16) \\
&= \sin 16
\end{aligned}$$

11. By the divergence theorem, we can calculate the integral of the divergence of $\mathbf{F}$ over the unit cube by calculating the flux of $\mathbf{F}$ out of the unit cube. But on the face $x = 0, 0 \le y, z \le 1$ of the unit cube, we see that the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ is $\langle -1, 0, 0\rangle$, while the x component of the vector $\mathbf{F}$ will always be zero (since when $x = 0$, $x(1-x)\log(1 + xyz)$ will always be zero) . Thus $\mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)$ is 0, and the flux integral over this face is zero. The same thing happens for all the other faces. Thus the answer is 0.

12. The answer is 1/96. You should have set up the following integrals; cartesians:

$$\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dx\,dy$$

cylindrical polars:

$$\int_0^1 \int_0^{\sqrt{1-r^2}} \int_0^{\pi/4} r^2 z\cos\theta\sin\theta r\,d\theta\,dz\,dr$$

spherical polars:

$$\int_0^1 \int_0^{\pi/2} \int_0^{\pi/4} \cos\theta\sin\theta\rho^5\sin^3\phi\cos\phi\,d\theta\,d\phi\,d\rho$$