MATH 20A — FINAL PRACTICE

The best way to prepare for a math exam is generally to do practice problems. Although the final problems will not follow homework problems as closely as the midterm did, it is still the case that doing homework problems from the book will be very good preparation for the exam: the majority of the final problems will be very similar to homework problems. There some harder problems too; these practice problems are meant to be examples of what you might expect.

1. Find the flux of the vector field $\langle 0, 0, x+z\rangle$ outward through the sphere $x^2+y^2+z^2=9$; do this both by calculating the integral directly, and by invoking Gauss's theorem.

2. Classify all critical points of the function $f(x,y) = 3xy + x^2y + xy^2$. Use this to draw a rough contour plot of the function.

3. Classify all critical points of the function $f(x,y) = y^3 - 3y^2 + 4x + x^2 - 3$. Use this to draw a rough contour plot of the function.

4. The three angles α, β, and γ in a triangle must, of course, add up to π. What is the maximum possible value of
$$\log(\sin(\alpha)\sin(\beta)\sin(\gamma))?$$

5. Evaluate the line integral of the vector field
$$\mathbf{F} = \langle e^{y^2} + z\cos(xz), 2xye^{y^2}, x\cos(xz)\rangle$$
along the curve given parametrically by:
$$\mathbf{r}(t) = \langle t\pi/2, 1-t, t^3\rangle$$
(for $0 \le t \le 1$).

6. We want to choose a point P somewhere on the line $x = y-1$ such that the quantity
$$\mathrm{dist}(P,A)^2 + 2\mathrm{dist}(P,B)^2$$
is minimized, where A is the point $(0,2)$ and B is the point $(0,0)$. What is the best possible choice of P?

7. Eustace and Hortense are as peculiar as their names would lead you to expect. Their idea of fun is playing the following 'game'. Eustace chooses two positive real numbers a and b, and Hortense then calculates the flux integral of
$$\mathbf{F} = \langle -x^2 - 4xy, -yz, 12z\rangle$$
outwards through the boundary of the solid in space defined by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le 1$. They want the answer to be as large as possible.

   Can you tell them what the best value for a and b would be?

8. Suppose a and b are positive lengths. We consider a solid cone in space, formed by the region above the plane $z = 0$ and below the graph of $\frac{z}{a} = 1 - \sqrt{\frac{x^2+y^2}{b^2}}$
   (a) Sketch the cone; what are the meanings of a and b?
   (b) Find a parameterization for the slanty section of the cone.
   (c) Find the surface area of this slanty section, and hence the total surface area of the cone.
   (d) Find the volume of the cone, by doing an appropriate integral in polar coordinates.
   (e) You are free to choose a and b as you like, as long as you ensure that the surface area is π. Use the method of Lagrange multipliers to find out how to choose a and b to maximize the volume of the cone. (Once you have found the critical point, you may assume it is a maximum without further work.)

9. Let $\mathbf{F}$ be the vector field
$$\mathbf{F}(x,y) = \langle -y^4/2 + y^3/6 - y, x^4 12 - x^3/6\rangle$$
and let $f(x,y)$ be the curl of $\mathbf{F}$. Find a formula for $f(x,y)$. Find the global maxima and minima of $f(x,y)$ inside the disc $x^2+y^2 \le 4$. (You will have to use IFR.)

10. Evaluate the following integral:

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz\cos(y^2)\,dy\,dx\,dz$$

You may have to think laterally!

11. Supposing $\mathbf{F}$ is the vector field

$$\langle x(1-x)\log(1+xyz), y(1-y)\tan(x^3+y^3+z^3), z(1-z)e^{\sqrt{x+y}}\rangle$$

what is the integral of the divergence of $\mathbf{F}$ over the unit cube $0 \leq x, y, z \leq 1$?

12. *This is maybe a little harder than a real final problem would be.* Integrate $f(x, y, z) = xyz$ over the solid in space described by the inequalities $0 \leq z \leq \sqrt{1-x^2-y^2}$, $x2 + y^2 \leq 1$, $x - y \geq 0$, $y \geq 0$. Do the integral fist in cartesians, then in cylindrical polars, then in spherical polars.