NOVEMBER 09 TBL — MATH 20A FINAL — FORMULA SHEET

- *Formula for curvature.* The formula is
$$\frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$
- *Arc length.* Formula: $\int_{t_0}^{t_1} |\mathbf{r}'(t)|\, dt$
- *Description of a plane.* A plane can be described in parametric form as
$$\mathbf{r}(u, v) = \mathbf{p} + u\mathbf{b} + v\mathbf{c}$$
and in implicit form as $\mathbf{r}.\mathbf{n} = c$. (Here $\mathbf{n}, c, \mathbf{p}, \mathbf{b}, \mathbf{c}$ stand for fixed numbers or vectors.)
- *Description of a plane.* A line can be described in parametric form as
$$\mathbf{r}(t) = \mathbf{p} + t\mathbf{b}$$
and most can be described in symmetric form as
$$\frac{x - *}{*} = \frac{y - *}{*} = \frac{z - *}{*}$$
where the $*$s stand for fixed numbers. (For some special lines, the symmetric form is more complicated; you will have to remember how this works.)
- *Distance from* **point** *to* **plane**.

  Let the point be $\mathbf{r}_P = (x_P, y_P, z_P)$. Let the plane be $\mathbf{n} \cdot \mathbf{r} = c$. The distance is
$$\frac{\mathbf{n} \cdot \mathbf{r}_P - c}{|\mathbf{n}|}$$
made into a positive quantity.
- *Distance from* **point** *to* **line**.

  Let the point be $\mathbf{r}_P = (x_P, y_P, z_P)$. Let the line be $\mathbf{r}(t) = \mathbf{r}_l + t\mathbf{a}_l$. The distance is
$$\frac{|(\mathbf{r}_P - \mathbf{r}_l) \times \mathbf{a}_l|}{|\mathbf{a}_l|}$$
which will always be a positive quantity.
- *Distance from* **line** *to* **line**.

  Let the line be $\mathbf{r}(t) = \mathbf{r}_l + t\mathbf{a}_l$. Let the other line be $\mathbf{r}(t) = \mathbf{r}_m + t\mathbf{a}_m$. The distance is
$$\frac{(\mathbf{r}_l - \mathbf{r}_m) \cdot (\mathbf{a}_l \times \mathbf{a}_m)}{|(\mathbf{a}_l \times \mathbf{a}_m)|}$$
made into a positive quantity.
- *Distance from* **plane** *to a* **parallel plane**.

  Let one plane be $\mathbf{n} \cdot \mathbf{r} = c$. You should be able to make the other plane into $\mathbf{n} \cdot \mathbf{r} = c'$. (If you can't, it isn't a *parallel* plane.)

  The distance is
$$\frac{c - c'}{|\mathbf{n}|}$$
made into a positive quantity.
- *2D Polar coordinates conversion formulae.*
$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \theta = \begin{cases} \arctan(x/y) \text{ or} \\ \arctan(x/y) + \pi \end{cases}$$
- *3D Polar coordinates conversion.* Convert cylindrical polars to cartesians using same process as in 2D (leave z coordinate alone). To interconvert spherical/cylinrical:
$$z = \rho\cos\phi \qquad \rho = \sqrt{z^2 + r^2}$$
$$r = \rho\sin\phi \qquad \phi = \arccos(z/\sqrt{z^2 + r^2})$$
and θ stays the same.
- *Chain rule example.* If f depends on x and y, each of which depends in turn on t, then
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
- *Gradient.* $\nabla f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle$
- *Tangent plane to graph of function.* Has equation
$$z = (x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\frac{\partial f}{\partial y} + z_0$$
- *Directional derivative.* $D_{\mathbf{u}} f = (\mathbf{u} \cdot \nabla f)/|\mathbf{u}|$
- *Classifying fixed points.* $D = (f_{xx} f_{yy}) - (f_{xy})^2$. If $D < 0$, saddle. If $D > 0$ and $f_{xx} > 0$, minimum. If $D > 0$ and $f_{xx} < 0$, maximum.
- *IFR for max/min.* (i) domain closed and bounded? (ii) Find all boundary pieces, and corners. (iii) Use $\nabla f = 0$ to find critical points in interior. (iv) Use single variable $f' = 0$ to find crit points on edge. (v) Work out function at corners. (vi) Whatever's the biggest/smallest, is the answer.

- *Method of Lagrange multipliers.* To find critical points of $f(x, y, z)$ subject to constraint that $g(x, y, z) = 0$, you should find critical points of

$$L(x, y, z, \lambda) := f(x, y, z) + \lambda g(x, y, z)$$

- *Surface area.* Formula $\int \int |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$
- *Integrating in polars.* Remember $dx \, dy \to r \, dr \, d\theta$. In 3D spherical polars $dx \, dy \, dz \to \rho^2 \sin\phi \, d\rho \, d\theta \, dr$.
- *Line integrals.* Formula

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

- *Fundamental theorem of line integrals.* If $\mathbf{F} = \nabla f$, and C is a path starting at $\mathbf{r}(t_0)$ and ending at $\mathbf{r}(t_1)$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0))$$

- *Flux integrals in 2D.* Formula

$$\int_C \mathbf{F} \cdot R d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot R\mathbf{r}'(t) \, dt$$

- *Surface integrals in 3D.* Formula

$$\int \int_R f dS = \int \int f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

- *Flux integrals in 3D.* Formula

$$\int \int_R \mathbf{F} \cdot d\mathbf{S} = \int \int \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$$

- *Div and curl in 2D.* If $\mathbf{F} = \langle \mathbf{F}_x, \mathbf{F}_y \rangle$

$$\text{div } \mathbf{F} = \frac{\partial \mathbf{F}_x}{\partial x} + \frac{\partial \mathbf{F}_y}{\partial y} \quad \text{curl } \mathbf{F} = \frac{\partial \mathbf{F}_y}{\partial x} - \frac{\partial \mathbf{F}_x}{\partial y}$$

  Div measures how much created or destroyed; curl measures how much momentum of vector field makes a little circle rotate counterclockwise.

- *Div and curl in 2D.* If $\mathbf{F} = \langle \mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z \rangle$

$$\text{div } \mathbf{F} = \frac{\partial \mathbf{F}_x}{\partial x} + \frac{\partial \mathbf{F}_y}{\partial y} + \frac{\partial \mathbf{F}_z}{\partial z}$$

$$\text{curl } \mathbf{F} = \langle \frac{\partial \mathbf{F}_z}{\partial y} - \frac{\partial \mathbf{F}_y}{\partial z}, \frac{\partial \mathbf{F}_x}{\partial z} - \frac{\partial \mathbf{F}_z}{\partial x}, \frac{\partial \mathbf{F}_y}{\partial x} - \frac{\partial \mathbf{F}_x}{\partial y} \rangle$$

  (so div $\mathbf{F} = \nabla \cdot \mathbf{F}$, curl $\mathbf{F} = \nabla \times \mathbf{F}$.) Div measures how much created or destroyed; curl measures how much momentum of vector field makes a little sphere rotate counterclockwise. (To translate rotations to vectors, use right hand grip.)

- *Green's theorem.* Line C goes around edge of region R, counterclockwise. Divergence form: flux outward through C = integral of divergence over R. Curl form: line integral around C, counterclockwise = integral of curl over R.
- *Gauss's theorem.* Surface S is boundary of 3D region R. Divergence form: flux outward through S = integral of divergence over R.
- *Stokes's theorem.* Curve C is boundary of 3D surface S. $\mathbf{F}$ is a vector field. Flux of curl $\mathbf{F}$ through $S = \pm$ line integral of $\mathbf{F}$ along C. (To get +, rather than -, apply right hand grip rule.)