Math 20A lecture 10

The Gradient Vector

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Announcements

- Homework five posted, due this Friday
- Office hours are 2–3.30pm today
- See the website for all sorts of course-related fun
  `http://people.brandeis.edu/~tbl/math20a/`
- It is *your responsibility* to log into LATTE and check that the grades are entered correctly. So far, HW 1–4 should be posted.
- Please give me feedback!
- I'm afraid your 'total score' in LATTE is not meaningful

Tangent planes to parametric surfaces

- Find an equation to the tangent plane of the parametric surface
$$\mathbf{r}(u, v) = \langle u^2, uv, v^3 \rangle$$
at the point corresponding to $u = 1, v = 2$.

Tangent planes to parametric surfaces

- Find an equation to the tangent plane of the parametric surface

$$\mathbf{r}(u,v) = \langle u^2, uv, v^3 \rangle$$

at the point corresponding to $u = 1, v = 2$.

- First we work out which point corresponds to $u = 1, v = 2$ by plugging in

$$\mathbf{r}(1,2) = \langle 1, 1, 8 \rangle$$

thus we know the tangent plane goes through this point.

Tangent planes to parametric surfaces

- Find an equation to the tangent plane of the parametric surface

$$\mathbf{r}(u,v) = \langle u^2, uv, v^3 \rangle$$

at the point corresponding to $u = 1, v = 2$.

- Now we consider u fixed at 1, and differentiate w.r.t. v to get the tangent vector to the curve $\mathbf{r}(v) = \mathbf{r}(1,v) = \langle 1, v, v^3 \rangle$

$$\mathbf{r}_v = \langle 0, 1, 3v^2 \rangle$$

Plugging in $v = 2$, we get $\mathbf{T} = \langle 0, 1, 12 \rangle$. *The tangent plane is parallel to this.*

Tangent planes to parametric surfaces

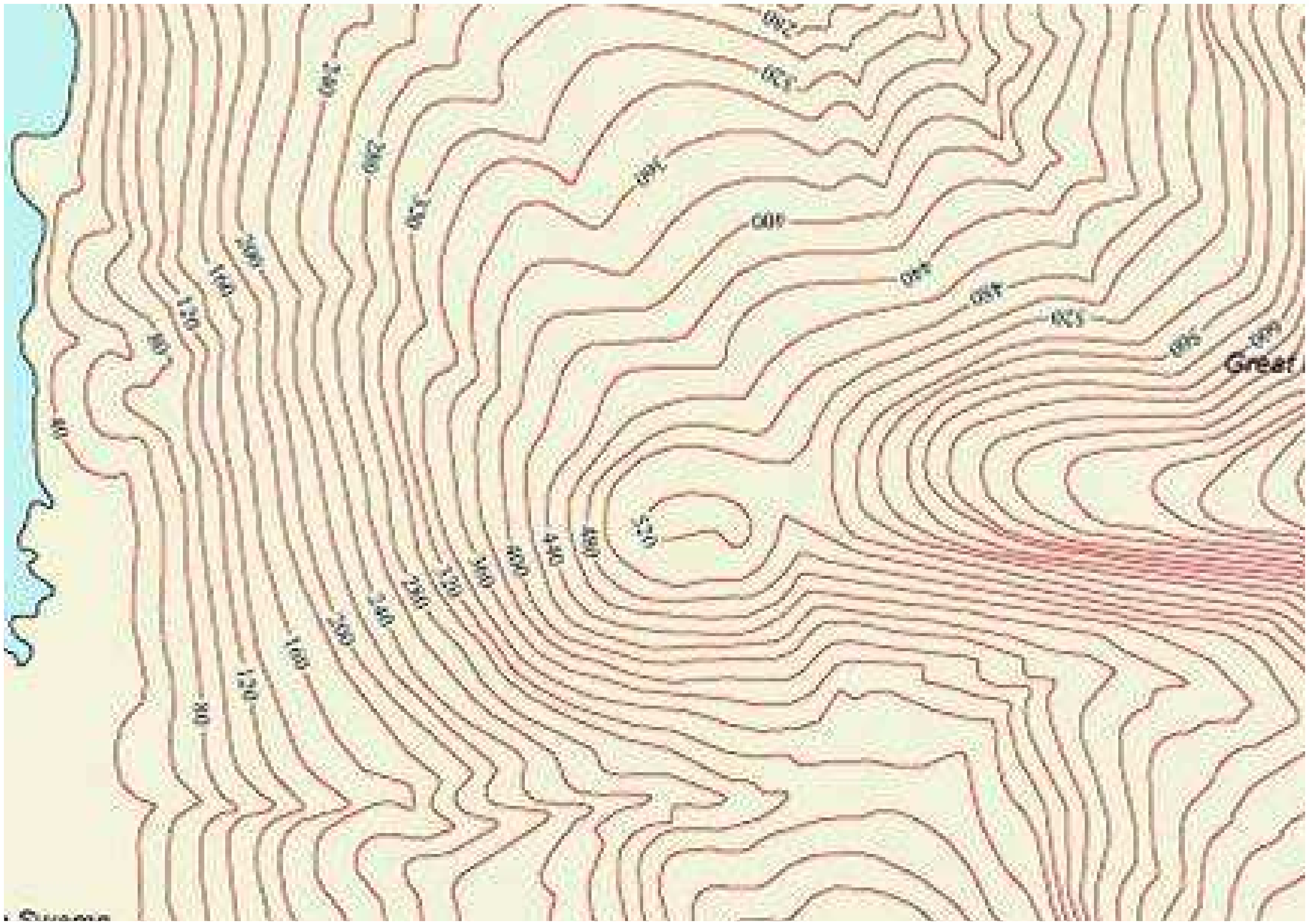
- Find an equation to the tangent plane of the parametric surface
$$\mathbf{r}(u,v) = \langle u^2, uv, v^3 \rangle$$
at the point corresponding to $u = 1, v = 2$.

- Now we consider v fixed at 2, and differentiate w.r.t. u to get the tangent vector to the curve
$\mathbf{r}(u) = \mathbf{r}(u,2) = \langle u^2, 2u, 8 \rangle$
$$\mathbf{r}_u = \langle 2u, 2, 0 \rangle$$
Plugging in $u = 1$, we get $\mathbf{T} = \langle 2, 2, 0 \rangle$. *The tangent plane is parallel to this.*

Tangent planes to parametric surfaces

- We want to find a plane through $(1, 2, 8)$, containing the vectors $\langle 0, 1, 12\rangle$ and $\langle 2, 2, 0\rangle$.
- The normal to the plane will be $\langle 0, 1, 12\rangle \times \langle 2, 2, 0\rangle$; that is, $\langle -24, 24, -2\rangle$. Thus we know the plane takes the form $\mathbf{n} \cdot \mathbf{r} = c$, or $-24x + 24y - 2z = c$. We just need to figure out c.
- We do this using the fact that $(1, 2, 8)$ is on the plane. We get $c = 8$. Thus the plane is $-24x + 24y - 2z = 8$.

Example: directional/partial derivatives from contours



Example: directional/partial derivatives from formula

Compute the directional derivative of the function

$$f(x, y) = xy$$

at the point $(1, -1)$ in the direction $\langle 1, 2 \rangle$.

Example: directional/partial derivatives from formula

Compute the directional derivative of the function

$$f(x, y) = xy$$

at the point $(1, -1)$ in the direction $\langle 1, 2 \rangle$.

- The partials are $f_x = y, f_y = x$.
- At $(1, -1)$, we get $f_x = -1, f_y = 1$.
- From the formula

$$D_{(1,2)}(f) = \frac{1f_x + 2f_y}{|\langle 1, 2 \rangle|} = \frac{-1 + 2}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

Example: directional/partial derivatives from formula II

Compute the directional derivative of the function

$$f(x,y) = x^2 + y^2 + y\sin x$$

at the point $(\pi, 0)$ in the direction $\langle 1, 1\rangle$.

Example: directional/partial derivatives from formula II

Compute the directional derivative of the function

$$f(x, y) = x^2 + y^2 + y \sin x$$

at the point $(\pi, 0)$ in the direction $\langle 1, 1 \rangle$.

- The partials are $f_x = 2x + y \cos x$, $f_y = 2y + \sin x$.
- At $(\pi, 0)$, we get $f_x = 2\pi$, $f_y = 0$.
- From the formula

$$D_{(1,1)}(f) = \frac{1 f_x + 1 f_y}{|\langle 1, 1 \rangle|} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Example: directional/partial derivatives from formula III

Compute the directional derivative of the function

$$f(x, y, z) = \sqrt{xyz}$$

at the point $(3, 2, 6)$ in the direction $\langle -1, -2, 2 \rangle$.

Example: directional/partial derivatives from formula III

Compute the directional derivative of the function

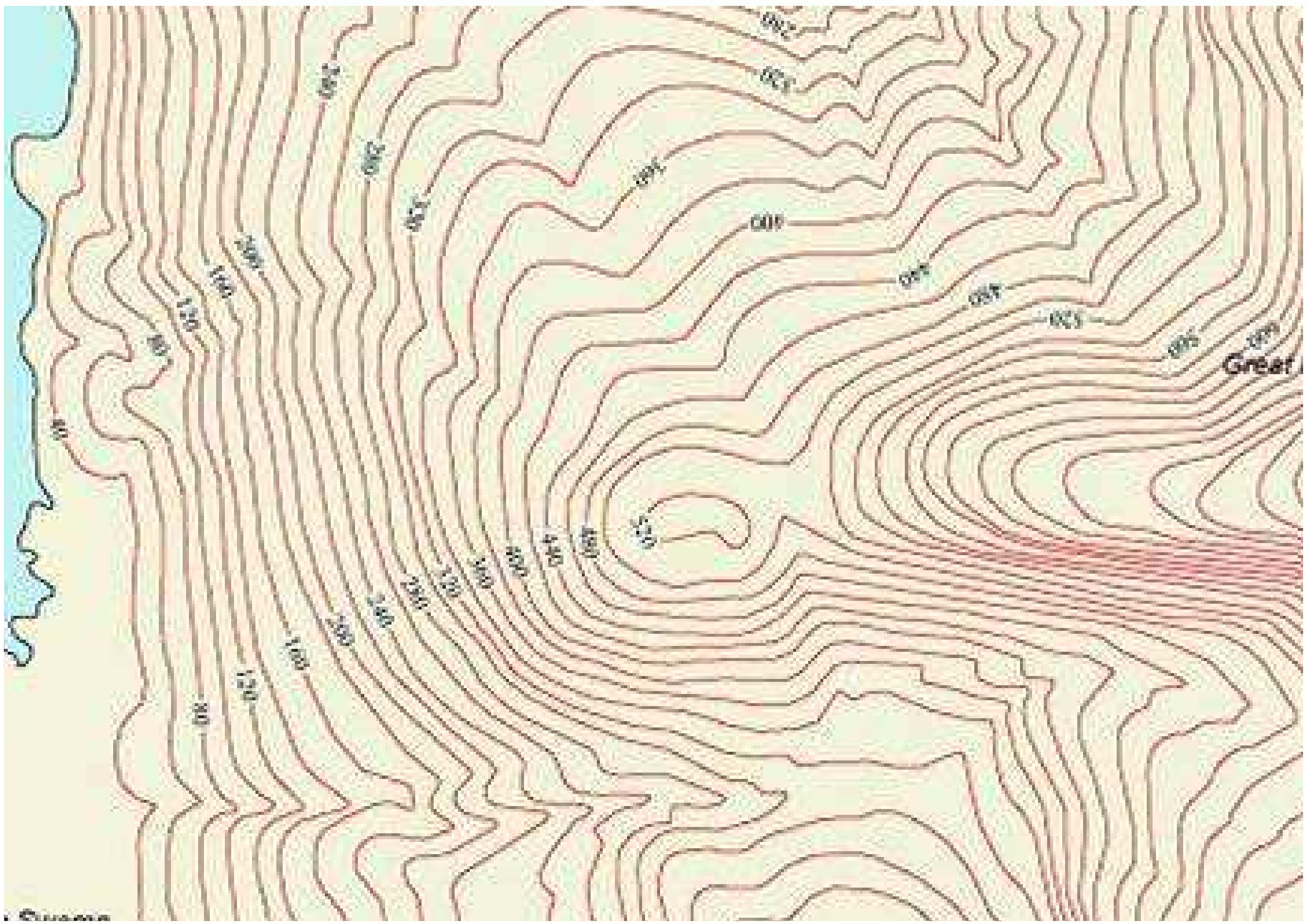
$$f(x, y, z) = \sqrt{xyz}$$

at the point $(3, 2, 6)$ in the direction $\langle -1, -2, 2\rangle$.

- The partials are $f_x = \sqrt{\frac{yz}{4x}}$, $f_y = \sqrt{\frac{xz}{4y}}$, $f_z = \sqrt{\frac{xy}{4z}}$.
- At $(3, 2, 6)$, we get $f_x = 1, f_y = (3/2), f_z = (1/2)$.
- From the formula

$$D_{(-1,-2,2)}(f) = \frac{-1f_x - 2f_y + 2f_z}{|\langle -1, -2, 2\rangle|} = \frac{-1-3+2}{\sqrt{9}} = -2/3$$

Example: gradient vectors from contours



Example: the gradient vector

Compute the maximum rate of change of the function

$$f(x, y) = x^2 + y^2 + y \sin x$$

at the point $(\pi, -\pi)$, and the direction in which it occurs.

Example: the gradient vector

Compute the maximum rate of change of the function

$$f(x,y) = x^2 + y^2 + y\sin x$$

at the point $(\pi, -\pi)$, and the direction in which it occurs.

- The partials are $f_x = 2x + y\cos x$, $f_y = 2y + \sin x$.
- At $(\pi, -\pi)$, we get $f_x = 2\pi - \pi$, $f_y = -2\pi$.
- The gradient vector ∇f is $(\pi, -2\pi)$
- The fastest rate of change is $|\nabla f| = \sqrt{5}\pi$
- The direction of fastest change is the direction $\langle \pi, -2\pi \rangle$. (A simpler vector in the same direction is $\langle 1, -2 \rangle$, and a unit vector is $\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$.)

Example: gradient vector II

Compute the maximum rate of change of the function

$$f(x, y, z) = \sqrt{x}\sin y + z$$

at the point $(2, \pi/2, 0)$, and the direction in which it occurs.

Example: gradient vector II

Compute the maximum rate of change of the function

$$f(x, y, z) = \sqrt{x}\sin y + z$$

at the point $(2, \pi/2, 0)$, and the direction in which it occurs.

- The partials are $f_x = \frac{\sin y}{2\sqrt{x}}$, $f_y = \sqrt{x}\cos y$, $f_z = 1$.
- At $(2, \pi/2, 0)$, we get $f_x = 1/4$, $f_y = 1$, $f_z = 1$.
- The gradient vector ∇f is $(1/4, 1, 1)$
- The fastest rate of change is $|\nabla f| = \sqrt{33}/4$
- The direction of fastest change is the direction $\langle 1/4, 1, 1\rangle$. (A simpler vector in the same direction is $\langle 1, 4, 4\rangle$, and a unit vector is $\langle \frac{1}{\sqrt{33}}, \frac{4}{\sqrt{33}}, \frac{4}{\sqrt{33}}\rangle$.)

Example: the tangent plane

Find the tangent plane to the implicit surface

$$x^2 + \cos y + z^3 = 2$$

at the point $(1, 0, 1)$

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Find the tangent plane to the implicit surface

$$x^2 + \cos y + z^3 = 2$$

at the point $(1, 0, 1)$

- We think of this as the level surface $f(x, y, z) = 2$ of the function $f(x, y, z) = x^2 + \cos y + z^3$
- The partials are $f_x = 2x$, $f_y = -\sin y$, $f_z = 3z^2$, so $\nabla f = \langle 2x, -\sin y, 3z^2 \rangle$.
- At the point $(1, 0, 1)$, this becomes $\nabla f = \langle 2, 0, 3 \rangle$.
- The tangent plane is therefore normal to $\langle 2, 0, 3 \rangle$. It passes through $(1, 0, 1)$. So plane is $2x + 3z = 5$.