Math 20A lecture 12

Taking things to extremes

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Announcements

- Homework six due Friday
- Office hours are 2–3.30pm today
- See the website for all sorts of course-related fun `http://people.brandeis.edu/~tbl/math20a/`
- It is *your responsibility* to log into LATTE and check that the grades are entered correctly. So far, HW 1–4 should be posted.

Previously on math 20a

- We talked about critical points, and saw how in the years BW we had to make do with finding these critical points when we wanted to sketch a function.
- We saw that, even in the years AM, this can be useful for finding formulae for maxes and mins...

Reminder: finding maxima exactly

I am a master horse whisperer, and I happen to know that two horses, Arm-n-a-leg and Buy-buy-baby, have a better than evens chance of winning their respective races. In particular, I know they both have a 0.7 chance of winning.

The bookies' are offering me even odds, so for every \$1000 I bet, I get \$2000 back if one of my horses wins, nothing back otherwise.

How much should I bet? (My current wealth W is \$10000, and my utility function is $\log(W/1000)$.)

Reminder: finding maxima exactly

Graphing the function
$U = 0.49\log(10+x+y) + 0.21\log(10+x-y) + 0.21\log(10-x+y) + 0.09\log(10-x-y)$ we see there's a single critical point which is the maximum we seek. We can find it by setting the partials to 0.

$$f_x = \frac{0.49}{10+x+y} - \frac{0.21}{10-x+y} + \frac{0.21}{10+x-y} - \frac{0.09}{10-x-y}$$
$$f_y = \frac{0.49}{10+x+y} + \frac{0.21}{10-x+y} - \frac{0.21}{10+x-y} - \frac{0.09}{10-x-y}$$

When $f_x = f_y = 0$,

$$0 = \frac{0.49}{10+x+y} - \frac{0.21}{10-x+y} + \frac{0.21}{10+x-y} - \frac{0.09}{10-x-y}$$

Reminder: finding maxima exactly

$$0 = \frac{0.49}{10+x+y} - \frac{0.21}{10-x+y} + \frac{0.21}{10+x-y} - \frac{0.09}{10-x-y}$$

$$0 = \frac{0.49}{10+x+y} + \frac{0.21}{10-x+y} - \frac{0.21}{10+x-y} - \frac{0.09}{10-x-y}$$

Subtracting, we get

$$0 = \frac{0.21}{10-x+y} - \frac{0.21}{10+x-y}$$

So $0.21(10+x-y) = 0.21(10-x+y)$, so $x = y$.

Example: finding maxima exactly

$$0 = \frac{0.49}{10+2x} - \frac{0.21}{10} + \frac{0.21}{10} - \frac{0.09}{10-2x}$$

So

$$0 = \frac{0.49}{10+2x} - \frac{0.09}{10-2x}$$

So $0 = (10-2x)0.49 - (10+2x)0.09$, so
$0 = 4.9 - 0.98x - 0.9 - 0.18x$, so $1.16x = 4$, so $x = 3.448$.
y is also 3.448.

Things that might happen

- There is no maximum: as you move out to infinity, you can keep improving things more and more, but never attain a single best result. (Only happens when domain *not bounded*.)
- There is no maximum: as you get closer to an excluded point on the edge, you can keep improving things more and more, but never attain a single best result. (Only happens when domain *not closed*.)
- There is a maximum on the inside of the domain. It will be a critical point.
- There is a maximum on the edge of the domain
- There is a maximum at a corner.

IFR

If you cannot visualize, proceed with utmost caution.

- Is the domain closed and bounded? (If not, either give up or visualize ...)
- Find all boundary pieces, and corners. (If you can't, you'll have to visualize ...)
- Use $\nabla f = 0$ to find critical points in interior.
- Use single variable $f' = 0$ to find crit points on edge.
- Work out function at corners.
- Whatever's the biggest/smallest, is the answer.
- *Note that you do not ever need to classify the fixed points as maxima/minima!*

Example: IFR

What are the maximum and minimum values of the function

$$5x + 5xy - x^2 - y^3$$

in the range $-1 \leq x, y \leq 3$?

Example: IFR

What are the maximum and minimum values of the function

$$5x + 5xy - x^2 - y^3$$

in the range $-1 \le x, y \le 3$?

- Note that the domain is closed and bounded
- Corners are $(-1, -1)$, $(-1, 3)$, $(3, -1)$, $(3, 3)$.
- Four boundary pieces: $x = -1$, $x = 3$, $y = -1$, $y = 3$.

Example: IFR

On the boundary piece $x = -1$, our function $f(x, y)$:

$$5x + 5xy - x^2 - y^3$$

becomes $g(y) = -5 - 5y - 1 - y^3$.

- $g'(y) = -5 - 3y^2$
- For critical points on boundary, $g'(y) = 0$, this has no real solutions.

Example: IFR

On the boundary piece $x = 3$, our function $f(x, y)$:

$$5x + 5xy - x^2 - y^3$$

becomes $g(y) = 15 + 15y - 9 - y^3$.

- $g'(y) = 15 - 3y^2$
- For critical points on boundary, $g'(y) = 0$, so $y = \pm\sqrt{5}$
- We want $-1 \leq x \leq 3$, so $y = \sqrt{5}$; then $f(3, \sqrt{5}) = 28.3607$.

Example: IFR

On the boundary piece $y = -1$, our function $f(x, y)$:

$$5x + 5xy - x^2 - y^3$$

becomes $g(x) = 5x - 5x - x^2 + 1 = 1 - x^2$.

- $g'(x) = -2x$
- For critical points on boundary, $g'(x) = 0$, this has solution $x = 0$
- $f(0, -1) = 1$

Example: IFR

On the boundary piece $y = 3$, our function $f(x, y)$:

$$5x + 5xy - x^2 - y^3$$

becomes $g(x) = 5x + 15x - x^2 - 27 = -27 + 20x - x^2$.

- $g'(x) = 20 - 2x$
- For critical points on boundary, $g'(x) = 0$, this has solution $x = 10$, which is outside the domain.

Example: IFR

On the interior, set $\nabla f = 0$ for our function $f(x, y)$:

$$5x + 5xy - x^2 - y^3$$

- $f_x = 5 + 5y - 2x$, $f_y = 5x - 3y^2$
- Solve these simultaneous equations (e.g. use the first to get x in terms of y, then put that into second to get quadratic in y, then for each y value solve back for x).
- $(x, y) = (5/12, -5/6)$ or $(15, 5)$. Only $(5/12, -5/6)$ is in range.

Example: IFR

$$f(-1,-1) = 0$$
$$f(3,-1) = -8$$
$$f(-1,3) = -48$$
$$f(3,3) = 24$$
$$f(0,-1) = 1$$
$$f(3,\sqrt{5}) = 28.3607$$
$$f(5/12,-5/6) = 0.752315$$

Thus the maximum is 28.36, the minimum is -48.
Don't you agree drawing a picture is awesome?

Example: constrained maximization

What is the largest possible volume for a box without a lid to be made from $12m^2$ of cardboard?

Example: constrained maximization

Maximize $x^2 + y^2 + 2z^2 + xy$ subject to the constraint that $x^2 + y^2 + z^2 - 1 = 0$.

Example: constrained maximization

What is the largest possible volume for a box without a lid to be made from $12m^2$ of cardboard?

- We can see intuitively that making any of the lengths close to zero is going to be a bad idea
- We can also see intuitively that making any of the lengths close to infinity is going to be a bad idea. (Because then another of the lengths $\to 0$.)
- Thus we know that there will be a maximum at a critical point.
- (N.B. on homework problems, I'll normally tell you you may assume that there is a maximum at a critical point, so you don't need to worry about writing this kind of argument out too much—I know expressing an intuitive argument can be painful.)

Example: constrained maximization

What is the largest possible volume for a box without a lid to be made from $12m^2$ of cardboard?

- We must find crit pts of $V(x, y, z) = xyz$ subject to $2xz + 2yz + xy = 12$.
- Let's write $A(x, y, z) = 2xz + 2yz + xy - 12$, so our constraint is $A(x, y, z) = 0$
- The method of Lagrange multipliers tells us to find the critical points of

  $$L(x, y, z, \lambda) = V(x, y, z) - \lambda A(x, y, z)$$

  That is:

  $$L(x, y, z, \lambda) = xyz - 2\lambda xz - 2\lambda yz - \lambda xy + 12\lambda$$

Example: constrained maximization

We're finding the critical points of

$$L(x, y, z, \lambda) = xyz + 2\lambda xz + 2\lambda yz + \lambda xy$$

We get the partials and set them to zero

$$yz - 2\lambda z - \lambda y = 0$$
$$xz - 2\lambda z - \lambda x = 0$$
$$xy - 2\lambda x - 2\lambda y = 0$$
$$2xz + 2yz + xy = 12$$

Solving, we get the answer $z = 1, x = 2, y = 2$.