

Math 20A lecture 14
More multivariate integration

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Announcements

- ⑥ Homework seven due Friday
- ⑥ Office hours are 2–3.30pm today
- ⑥ See the website for all sorts of course-related fun
<http://people.brandeis.edu/~tbl/math20a/>
- ⑥ It is *your responsibility* to log into LATTE and check that the grades are entered correctly. So far, HW 1–5 should be posted.

Previously on math 20a

- ⑥ We talked about extending the concept of integration to functions of two variables; in particular, we wanted to compute the volume under the graph of a function of two variables.
- ⑥ As usual, once we can work out how to visualize things, the actual calculation rapidly reduces to things you already know from single variable—in this case, integrating a function of one variable.
- ⑥ The difference is that now you must do *two* one variable integrations in quick succession, one for the x coordinate and one for the y
- ⑥ This is justified by thinking about slicing

Previously on math 20a

- ⑥ We reasoned that since you can compute the same volume by slicing in the x direction or by slicing in the y direction, these two calculations must give the same answer.
- ⑥ But we also saw that even though you get the same answer either way, the calculations can be a lot less pleasant one way than the other, so it pays to strategize carefully about which way to do things.
- ⑥ Finally, we started to look at non-rectangular regions...

Example: Integrating over interesting regions

What is the volume under the graph of $f(x, y) = x + 2y$ above the region between the curves $y = 2x^2$ and $y = 1 + x^2$?


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- ⑥ First we draw a picture...
- ⑥ We see it is easier to slice parallel to the y axis.
- ⑥ At each slice of constant x , the *lowest* possible y value is $2x^2$ and the *highest* possible is $1 + x^2$.
- ⑥ We must make a slice for each x between -1 and 1 . We get the following iterated integral:

$$\int_{-1}^1 \int_{2x^2}^{1+x^2} x + 2y \, dy \, dx$$

Example: Integrating over interesting regions


$$\begin{aligned}\int_{-1}^1 \int_{2x^2}^{1+x^2} x + 2y \, dy \, dx &= \int_{-1}^1 \left[xy + y^2 \right]_{y=2x^2}^{1+x^2} dx \\ &= \int_{-1}^1 \left[x(1+x^2) + (1+x^2)^2 \right. \\ &\quad \left. - [x(2x^2) + (2x^2)^2] \right] dx \\ &= \int_{-1}^1 1 + x + 2x^2 - x^3 - 3x^4 \, dx \\ &= \left[x + \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} - \frac{3x^5}{5} \right]_{-1}^1 \\ &= 32/15\end{aligned}$$

Example: Integrating in polars

What is the volume under the graph of $f(x, y) = 1 - x^2 - y^2$ above the region $x^2 + y^2 \leq 1$? (Hint: use polars!)

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⑥ The volume is

$$\begin{aligned}\int_0^1 \int_0^{2\pi} (1 - r^2)r \, d\theta \, dr &= \int_0^1 2\pi(1 - r^2)r \, dr \\ &= \int_0^1 2\pi r - 2\pi r^3 \, dr \\ &= \left[\frac{2\pi r^2}{2} - \frac{2\pi r^4}{4} \right]_0^1 \\ &= \pi - \frac{\pi}{2} = \pi/2\end{aligned}$$

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What is the volume under the graph of $f(x, y) = 3x + 4y^2$ above the region $1 \leq x^2 + y^2 \leq 4$? (Hint: use polars!)

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- ⑥ The integrand can be expressed in polars as

$$f = 3r \cos \theta + 4r^2 \sin^2 \theta.$$

- ⑥ The volume is

$$\begin{aligned} & \int_1^2 \int_0^{2\pi} (3r \cos \theta + 4r^2 \sin^2 \theta) r \, d\theta \, dr \\ &= \int_1^2 \int_0^{2\pi} 3r^2 \cos \theta + 4r^3 \sin^2 \theta \, d\theta \, dr \\ &= \int_1^2 3r^2 \left(\int_0^{2\pi} \cos \theta \, d\theta \right) + 4r^3 \left(\int_0^{2\pi} \sin^2 \theta \, d\theta \right) \, dr \end{aligned}$$

Example: Integrating in polars

Continuing...

$$\begin{aligned} & \int_1^2 3r^2 \left(\int_0^{2\pi} \cos \theta d\theta \right) + 4r^3 \left(\int_0^{2\pi} \sin^2 \theta d\theta \right) dr \\ &= \int_1^2 4r^3 \pi dr = \left[r^4 \pi \right]_1^2 \\ &= (2^4 \pi - 1^4 \pi) = 15\pi \end{aligned}$$

The book gives a similar example, but in the book they only do the integral over the part of the region $1 \leq x^2 + y^2 \leq 4$ in the upper half plane, so they take $0 \leq \theta \leq \pi$ and get a different answer.