Math 20A lecture 15

Volume integrals and surface area

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Announcements

- Homework seven due today.
- Homework eight will be posted soon, due 1 week from today.
- Office hours are 3–4.30pm today
- See the website for all sorts of course-related fun `http://people.brandeis.edu/~tbl/math20a/`
- It is *your responsibility* to log into LATTE and check that the grades are entered correctly. So far, HW 1–5 should be posted.

Previously on math 20a

- We talked about extending the concept of integration to functions of two variables; in particular, we wanted to compute the volume under the graph of a function of two variables.
- (We also talked a little about polar coordinates... but we wont be thinking about them today.)

Example: Integrals without volumes

The land value (in dollars per square foot) in a certain city can be modelled as

$$f(x, y) = 10 - x^2 - y^2$$

where we think of the city as being mapped out in the xy plane as $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.
What is the total land value?

Example: Integrals without volumes

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where we think of the city as being mapped out in the xy plane as $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.
What is the total land value?

$$\int_{-1}^{1} \int_{-1}^{1} 10 - x^2 - y^2 \, dx \, dy$$

Example: 3D Integrals

A smoke plume from a factory has risen into the air space above it. The plume is confined to a cuboid above the factory, which we can describe in coordinates as $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $0 \leq z \leq 1$. The concentration of toxic particles (in mols per m^3) can be modeled by:

$$c(x, y, z) = (x + 1)(x - 1)(y + 1)(y - 1)(1 - z)$$

How might we figure out the total amount of toxic particles discharged? (‘Amount’ means ‘number of moles’).

Example: 3D Integrals

The concentration of toxic particles (in mols per m^3) can be modeled by:

$$c(x, y, z) = (x+1)(x-1)(y+1)(y-1)(1-z)$$

How might we figure out the total amount of toxic particles discharged?

Example: 3D Integrals

The concentration of toxic particles (in mols per m^3) can be modeled by:

$$c(x, y, z) = (x + 1)(x - 1)(y + 1)(y - 1)(1 - z)$$

How might we figure out the total amount of toxic particles discharged?

- It is just a matter of generalizing the previous answer to three dimensions. We get $\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1} c(x, y, z)\, dz\, dy\, dx$ or

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1}(x + 1)(x - 1)(y + 1)(y - 1)(1 - z)\, dz\, dy\, dx$$

Example: 3D regions

The chemistry annex of my old school was an irregularly shaped space, which can be thought of as the region bounded by

$$z = 0, z = 2 + x + y, x = -1, x = 1, y = 0, y = 1 - x^2$$

There is a noxious vapor in the room, and its concentration is given by $f(x, y, z) = y^2$.
Write down an iterated integral for the total amount of toxious noxin.

Example: 3D regions

- First we draw a picture.
- We see that it's a good idea to slice in the x, then y, then z directions.
- We get

$$\int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2+x+y} y^2 \, dz \, dy \, dx$$

Exercise: 3D regions

Working with a partner, write the integral of the function $f(x, y, z)$ over the region between $z = 0$ and $z = 1 - x^2 - y^2$ as an iterated integral in as many orders as you can.

Example: Surface area

What is the surface area of the sphere with parametric description

$$\mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$$

for $0 \le u \le \pi$, $0 \le v \le 2\pi$?

Example: Surface area

What is the surface area of the sphere with parametric description

$$\mathbf{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$$

for $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$?

- We work out the partials

$$\mathbf{r}_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$$
$$\mathbf{r}_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$$

- We work out the cross product of these

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \rangle$$

Example: Surface area

And then the magnitude of the cross product

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \sin^2 u \cos^2 u}$$
$$= \sqrt{\sin^4 u + \sin^2 u \cos^2 u} = \sqrt{\sin^2 u(\sin^2 u + \cos^2 u)}$$
$$= \sqrt{\sin^2 u} = \sin u$$

So surface area is:

$$\int_0^{\pi} \int_0^{2\pi} \sin u \, dv \, du = \int_0^{\pi} 2\pi \sin u \, du$$
$$= [-2\pi \cos u]_0^{\pi} = 4\pi$$

Example: Surface area

What is the surface area of the surface with parametric description

$$\mathbf{r}(u, v) = \langle u \sin v, u \cos v, 1 - u^2 \rangle$$

for $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$?

Example: Surface area

What is the surface area of the surface with parametric description

$$\mathbf{r}(u, v) = \langle u \sin v, u \cos v, 1 - u^2 \rangle$$

for $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$?

- We work out the partials

$$\mathbf{r}_u = \langle \sin v, \cos v, -2u \rangle$$
$$\mathbf{r}_v = \langle u \cos v, -u \sin v, 0 \rangle$$

- We work out the cross product of these

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u^2 \sin v, -2u^2 \cos v, -u \rangle$$

Example: Surface area

And then the magnitude of the cross product

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4u^4 \sin^2 v + 4u^4 \cos^2 v + u^2}$$
$$= \sqrt{4u^4 + u^2}$$

So surface area is:

$$\int_0^1 \int_0^{2\pi} \sqrt{4u^4 + u^2}\, dv\, du = \int_0^1 (2\pi)\sqrt{4u^4 + u^2}\, du$$
$$= \int_0^1 (2\pi) u\sqrt{4u^2 + 1}\, du$$
$$= \int_1^5 \frac{2\pi}{8}\sqrt{t}\, dt = \ldots$$