Math 20A lecture 18

The fundamental theorem of line integrals

T.J. Barnet-Lamb

tbl@brandeis.edu

Brandeis University

Announcements

- Homework nine due Friday.
- Office hours are 2–3.30pm today
- See the website for all sorts of course-related fun `http://people.brandeis.edu/~tbl/math20a/`
- It is *your responsibility* to log into LATTE and check that the grades are entered correctly. So far, HW 1–6 should be posted.

Previously on math 20a

- We looked at vector fields in two dimensions, and tried to practice visualizing them.
- We looked vector line integrals, which are very important. We can visualize what they look like by imagining ‘freezing’ everything except a small tube of a vector field, and asking how much ‘net’ momentum along the tube there is.
- We talked about the gradient vector field of a function.

Example: recognizing conservatism

The vector field

$$\mathbf{F}(x, y) = \langle y^2, 2xy + 1 \rangle$$

is conservative. Find a potential (a function f with $\nabla f = \mathbf{F}$).

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- Considering the x component of $\mathbf{F}$ tells us that

  $$f(x,y) = xy^2 + h(y)$$

  for some function h of y. Then looking at the y component of $\mathbf{F}$ tells us $h_y(y) = 1$, so $h(y) = y + c$, for c any constant. Thus we can take (e.g.)

  $$f(x,y) = xy^2 + y$$

Example: recognizing non-conservatism

Show that the vector field

$$\mathbf{F}(x, y) = \langle y^2, 2xy + x \rangle$$

cannot be conservative

Example: recognizing non-conservatism

Show that the vector field

$$\mathbf{F}(x, y) = \langle y^2, 2xy + x \rangle$$

cannot be conservative

- If it were, then the x derivative of the y component should match the y derivative of the x component. But

$$\frac{\partial(2xy + x)}{\partial x} = 2y + 1$$

$$\frac{\partial(y^2)}{\partial y} = 2y$$

so the vector field is not conservative.

Example: recognizing non-conservatism

Show that the vector field

$$\mathbf{F}(x, y, z) = \langle y^2, 2xy + 1, zy \rangle$$

cannot be conservative

Example: recognizing non-conservatism

Show that the vector field

$$\mathbf{F}(x, y, z) = \langle y^2, 2xy + 1, zy \rangle$$

cannot be conservative

- If it were, then
  - the x derivative of the y component should match the y derivative of the x component, and
  - the z derivative of the y component should match the y derivative of the z component, and
  - the x derivative of the z component should match the z derivative of the x component.
- But $\frac{\partial(y^2)}{\partial z} = 0 \neq \frac{\partial(zy)}{\partial y} = 2y$ so the vector field is not conservative.

WARNING

There are some vector fields which are not gradient vector fields, but which the incompatible derivatives test will not detect. That is, these devious vector fields satisfy

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

but are still not the gradient of any function.

We will return to this topic in due course to explain why, but for now don't worry—for now, vector fields will always either be gradients or will be detected by the test.

If you're the kind of person who likes reading the last page of the murder mystery first, the book semi-explains more...

Example: using the fundamental theorem

Remember that

$$\mathbf{F}(x, y) = \langle y^2, 2xy + 1\rangle$$

is conservative, being the gradient vector field of $f(x, y) = xy^2 + y$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the path parameterized by $\mathbf{r}(t) = \langle t, t^2 + \cos t(\frac{\sqrt{t^{1000}+33t^4+100}}{t^2+1})\rangle$ for $-\pi/2 \leq t \leq \pi/2$

Example: using the fundamental theorem

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is conservative, being the gradient vector field of $f(x,y) = xy^2 + y$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the path parameterized by $\mathbf{r}(t) = \langle t, t^2 + \cos t(\frac{\sqrt{t^{1000}+33t^4+100}}{t^2+1})\rangle$ for $-\pi/2 \le t \le \pi/2$

Because of the fundamental theorem, only the *endpoints* of the path matter. Plugging in $t = -\pi/2$, $t = \pi/2$, we get that the path goes from $(-\pi/2, \pi^2/4)$ to $(\pi/2, \pi^2/4)$, and so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(\pi/2, \pi^2/4) - f(-\pi/2, \pi^2/4) = (\pi^5/32 + \pi^2/4) - (-\pi^5/32 + \pi^2/4) = \pi^5/16$

Exercises

1. Which of the following two vector fields is conservative? Find a potential for the conservative one.

$$\mathbf{F}_1(x, y) = \langle \sin y, x \cos y \rangle \quad \mathbf{F}_2(x, y) = \langle \sin y, x \sin y \rangle$$

2. The vector field

$$\mathbf{F}(x, y, z) = \langle y^2 z, 2xyz + z, xy^2 + y \rangle$$

is conservative. Find a potential for $\mathbf{F}$, and compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the path parameterized by $\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle$ for t between 0 and 2π.

Example: computing a 2D flux integral

Compute the flux of the vector field

$$\mathbf{F}(x, y) = \langle xy^2, y^3 \rangle$$

through the closed curve parameterized by

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle$$

for $0 \leq t \leq 2\pi$.

Example: computing a 2D flux integral

We work out

$$\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$$
$$R\mathbf{r}'(t) = \langle \cos t, \sin t \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \langle (\cos t)(\sin t)^2, (\sin t)^3 \rangle$$

And the flux integral is

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot R\mathbf{r}'(t)\, dt = \int_0^{2\pi} (\cos t)^2 (\sin t)^2 + (\sin t)^4 \, dt$$
$$= \int_0^{2\pi} (\sin t)^2 \, dt = \pi$$

Example: computing a 2D flux integral

Compute the flux of the vector field

$$\mathbf{F}(x, y) = \langle 0, y \rangle$$

through the closed curve parameterized by

$$\mathbf{r}(t) = \langle 2t + 1, 0 \rangle$$

for $-1 \leq t \leq 0$, and by

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle$$

for $0 \leq t \leq \pi$.

Example: computing a 2D flux integral

- We must do each part of the curve separately. For the first part...

$$\mathbf{r}'(t) = \langle 2, 0 \rangle$$
$$R\mathbf{r}'(t) = \langle 0, -2 \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \langle 0, 0 \rangle$$

- And the flux integral is

$$\int_{-1}^{0} \mathbf{F}(\mathbf{r}(t)) \cdot R\mathbf{r}'(t)\, dt = \int_{-1}^{0} 0\, dt = 0$$

Example: computing a 2D flux integral

- We must do each part of the curve separately. For the first part...

$$\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$$
$$R\mathbf{r}'(t) = \langle \cos t, \sin t\rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \langle 0, \sin t\rangle$$

- And the flux integral is

$$\int_0^\pi \mathbf{F}(\mathbf{r}(t)) \cdot R\mathbf{r}'(t)\, dt = \int_0^\pi \sin^2 t = \pi/2$$

- Add both parts to get final answer: $0 + \pi/2 = \pi/2$