Math 20A lecture 20

Integral theorems

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Announcements

- Homework ten due Friday.
- Office hours are 2–3.30pm today
- See the website for all sorts of course-related fun `http://people.brandeis.edu/~tbl/math20a/`
- It is *your responsibility* to log into LATTE and check that the grades are entered correctly. So far, HW 1–6 and HW 8 should be posted.

Previously on math 20a

- We looked at vector fields in two dimensions.
- We looked at several kinds of line integrals. The two most important ones were *vector line integrals* and *flux integrals*.
- Vector line integrals measure how much the vector field is helping/hindering us as we try to move *along* a path
- Flux integrals measure how much ‘vector stuff’ is going *through* a boundary.
- We looked at div (which measures how much ‘vector stuff’ is being created or destroyed at a given point in a static flow, and curl (which measures how much a little circle/sphere would want to rotate from the momentum the flow gives it.

Reminder example: computing a 3D curl

Compute the curl of the vector field

$$\mathbf{F} = \langle z\cos x, \sin x + y, xyz\rangle$$

Reminder example: computing a 3D curl

Compute the curl of the vector field

$$\mathbf{F} = \langle z\cos x, \sin x + y, xyz\rangle$$

We just use the rule

$$\begin{aligned}\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \langle \frac{\partial \mathbf{F}_z}{\partial y} - \frac{\partial \mathbf{F}_y}{\partial z}, \frac{\partial \mathbf{F}_x}{\partial z} - \frac{\partial \mathbf{F}_z}{\partial x}, \frac{\partial \mathbf{F}_y}{\partial x} - \frac{\partial \mathbf{F}_x}{\partial y}\rangle \\ &= \langle yz - 0, \cos x - xy, \cos x - 0\rangle\end{aligned}$$

Example: Green’s theorem

Compute the flux of the vector field $\mathbf{F} = \langle x, y\rangle$ through the circle parameterized by $\mathbf{r}(t) = \langle\cos(t), \sin(t)\rangle$ both directly and using Green’s theorem.

Example: Green’s theorem

Directly. We work out

$$\begin{aligned} \mathbf{r}'(t) &= \langle -\sin t, \cos t\rangle \\ R\mathbf{r}'(t) &= \langle \cos t, \sin t\rangle \\ \mathbf{F}(\mathbf{r}(t)) &= \langle \sin t, \cos t\rangle \end{aligned}$$

And the flux integral is

$$\begin{aligned} \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot R\mathbf{r}'(t)\, dt &= \int_0^{2\pi} (\cos t)^2 + (\sin t)^2\, dt \\ &= \int_0^{2\pi} 1\, dt = 2\pi \end{aligned}$$

Example: Green's theorem

Using Green. We evaluate

$$\text{div}\,\mathbf{F} = \partial_x \mathbf{F}_x + \partial_y \mathbf{F}_y = 1 + 1 = 2$$

We then need to compute the integral of the constant function 2 over the unit circle. But this is just $2 \times ($area of circle$)$ =2π.

(If we'd wanted to do a formal integral, we'd do it in polars, getting

$$\int_R 2\,dx\,dy = \int_0^{2\pi}\int_0^1 2\,r\,dr\,d\theta \quad = \int_0^{2\pi} r^2|_0^1\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi \text{)}$$

Example: Green’s theorem II

Compute the flux of the vector field $\mathbf{F} = \langle x + \sin\sin y, \cos\cos x \rangle$ through the circle parameterized by $\mathbf{r}(t) = \langle \sin(t), \cos(t) \rangle$

Example: Green's theorem II

Compute the flux of the vector field $\mathbf{F} = \langle x + \sin\sin y, \cos\cos x \rangle$ through the circle parameterized by $\mathbf{r}(t) = \langle \sin(t), \cos(t) \rangle$

If we tried to do this directly, we'd end up having to integrate $\sin\sin t$, which is not possible.

Example: Green's theorem II

Compute the flux of the vector field $\mathbf{F} = \langle x + \sin\sin y, \cos\cos x\rangle$ through the circle parameterized by $\mathbf{r}(t) = \langle \sin(t), \cos(t)\rangle$

If we tried to do this directly, we'd end up having to integrate $\sin\sin t$, which is not possible.

But this integral is easy using Green's theorem. The divergence is 1+0=1, and the integral of 1 over the unit circle is π.

Example: Green’s theorem (curl form)

Compute the line integral of the vector field

$$\mathbf{F} = \langle -y, x \rangle$$

around the closed curve parameterized by $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, both directly and using Green’s theorem.

Example: Green’s theorem (curl form)

Compute the line integral of the vector field

$$\mathbf{F} = \langle -y, x\rangle$$

around the closed curve parameterized by $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$, both directly and using Green’s theorem.
Using Green. We calculate

$$\text{curl } \mathbf{F} = \partial_x \mathbf{F}_y - \partial_y \mathbf{F}_x = 1 - (-1) = 2$$

and we get the answer by integrating this over the unit circle, which gives us 2π.

Example: Green's theorem (curl form)

(Remember $\mathbf{F} = \langle -y, x \rangle$, $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$.)

Directly. We compute

$$\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle$$

And we use the formula...

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$

Exercise: Green's theorem (curl form)

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^2 + xy, y^2 + 2y \rangle$, and

$$\mathbf{r}(t) = \begin{cases} \langle 1, t \rangle & (0 \leq t \leq 1) \\ \langle 2 - t, 1 \rangle & (1 \leq t \leq 2) \\ \langle 0, 3 - t \rangle & (2 \leq t \leq 3) \\ \langle t - 3, 0 \rangle & (3 \leq t \leq 4) \end{cases}$$

using Green's theorem, and contemplate doing it without. (Hint: start by visualizing the path...)