Math 20A lecture 23

More on integral theorems

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Announcements

- Homework eleven due today.
- Homework twelve is posted, due Tuesday after Thanksgiving.
- Office hours are 3–4.30pm today
- See the website for all sorts of course-related fun `http://people.brandeis.edu/~tbl/math20a/`
- It is *your responsibility* to log into LATTE and check that the grades are entered correctly. So far, HW 1–6 and HW 8–10 should be posted.

Previously on math 20a

- We looked at several kinds of integral in 2D/3D.
  - Vector line integrals measure how much the vector field is helping/hindering us as we try to move *along* a path
  - Flux integrals measure how much ‘vector stuff’ is going *through* a boundary (which will be a line in 2D, a surface in 3D).
- We looked at
  - div (which measures how much ‘vector stuff’ is being created or destroyed at a given point in a flow), and
  - curl (which measures how much a little circle/sphere would want to rotate from the momentum the flow gives it).

Previously on math 20a

- We saw two forms of Greens' theorem (not counting the crazy one in the book!)
  - One connected *2D flux integrals* with (2D region) integrals of div.
  - One connected *2D line integrals* with (2D region) integrals of curl.
- And we saw their counterparts in 3D:
  - Gauss's theorem (AKA the divergence theorem) connects *3D surface flux integrals* with (2D region) integrals of div.
  - Stokes's theorem connects *3D line integrals* with *3D surface flux integrals* integrals of curl.

Example: Stokes's theorem (review)

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circular path $\mathbf{p}(t) = \langle \cos t, \sin t, 1 + \sin t + \cos t \rangle$ and $\mathbf{F}$ is the vector field $\mathbf{F}(x, y, z) = \langle -y, x^2 - x, z^2 - z + \cos\cos\cos z \rangle$.

Example: Stokes's theorem (review)

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circular path $\mathbf{p}(t) = \langle \cos t, \sin t, 1 + \sin t + \cos t \rangle$ and $\mathbf{F}$ is the vector field $\mathbf{F}(x, y, z) = \langle -y, x^2 - x, z^2 - z + \cos\cos\cos z \rangle$.
The curl is simply $\nabla \times \mathbf{F} = \langle 0, 0, 2x \rangle$. Thus we just need to integrate the flux of $\langle 0, 0, 2x \rangle$ through any surface with boundary C.

Let's take the elliptical region in the plane $z = 1 + x + y$.
This has parameterization

$$\mathbf{r}(u, v) = \langle v \cos u, v \sin u, 1 + v \cos u + v \sin u \rangle$$

for $0 \leq 0 \leq 2\pi$ and $0 \leq v \leq 1$.

Example: Stokes's theorem (review)

We work out

$$(\nabla \times \mathbf{F})(\mathbf{r}(u,v)) = \langle 0, 0, 2v\cos u\rangle$$
$$\mathbf{r}_u = \langle -v\sin u, v\cos u, -v\sin u + v\cos u\rangle$$
$$\mathbf{r}_v = \langle \cos u, \sin u, \cos u + \sin u\rangle$$
$$\mathbf{r}_u \times \mathbf{r}_v = \langle ?, ?, -v\sin^2 u - v\cos^2 u\rangle = \langle ?, ?, -v\rangle$$

We want the flux upwards through the surface, so we want the *other* normal. Then the answer is

$$\int_0^{2\pi}\int_0^1 \mathbf{F}(\mathbf{r}(u,v)) \cdot (-\mathbf{r}_u \times \mathbf{r}_v)\, dv\, du = \int_0^{2\pi}\int_0^1 2v^2\cos u\, dv\, du$$

Example: Stokes's theorem (review)

Working out the integral

$$\int_0^{2\pi}\int_0^1 \mathbf{F}(\mathbf{r}(u,v))\cdot(-\mathbf{r}_u\times\mathbf{r}_v)\,dv\,du = \int_0^{2\pi}\int_0^1 2v^2\cos u\,dv\,du$$

$$= \int_0^{2\pi}\frac{2}{3}v^3\Big|_0^1\cos u\,du = \int_0^{2\pi}\frac{2}{3}\cos u\,du = 0$$

Example: Stokes's theorem II

Stokes's theorem can be useful in the other direction too!

Compute the surface integral measuring the flux passing upwards through the upper half of the unit sphere of the vector field for the vector field

$$\mathbf{G} = \langle \cos(y) - \cos(z), -1, 4x^2 \rangle.$$

[Hint: $\mathbf{G}$ can be expressed as the curl of another vector field $\mathbf{F}$...

$$\mathbf{G} = \text{curl } \mathbf{F} = \text{curl } \langle -yx^2, x^3 + \sin(z), \sin(y) \rangle.]$$

Example: Stokes's theorem II

By Stokes's theorem, we can work out the answer by working out the line integral of $\mathbf{F}$ around the boundary of our surface. This boundary is the unit circle in the xy plane, and may be parameterized as:

$$\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$$

(For $0 \leq t \leq 2\pi$.) Check, using right hand rule, that we're going round the right way.

Then

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \langle -(\sin t)(\cos t)^2, (\cos t)^3, \sin \sin t \rangle$$

Example: Stokes's theorem II

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \langle -(\sin t)(\cos t)^2, (\cos t)^3, \sin\sin t\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (\sin t)^2(\cos t)^2 + (\cos t)^4 = (\cos t)^2$$

Then

$$\text{answer} = \iint_S \mathbf{G} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S}$$

$$= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} (\cos t)^2 = \pi$$

Stokes's theorem III

Exercise: Compute the surface integral measuring the flux passing upwards through the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$, and above the xy plane, of the vector field $\mathbf{G}$, where $\mathbf{G}$ is the curl of the vector field $\mathbf{F}$:

$$\mathbf{F} = \langle xz, yz, xy \rangle$$

Stokes's theorem III

First we draw a picture. (And orient ourselves.)

Stokes's theorem III

We must first parameterize the boundary of the surface. Visually, we see that the intersection will be a circle at constant height. We can also see this algebraically by subtracting the two equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$, which gives us that $z^2 = 3$, so $z = \sqrt{3}$ (remember $z > 0$).

Thus we're parameterizing a circle in the plane $z = \sqrt{3}$ of radius 1. (Because the cylinder had radius 1.) We get:

$$\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$$

(For $0 \leq t \leq 2\pi$.)

Stokes's theorem III

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t \sin t\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$$

Then:

$$\text{answer} = \iint_S \mathbf{G} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S}$$

$$= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0$$