# Math 20A lecture 3

# Lines and curves

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## *Announcements*

- Homework due now, accepted until 10.45am
- A new homework will been posted on the class website this afternoon, due Sept 11th at the beginning of class
- Office hours are 3–4.30pm today
- See the website for all sorts of course-related fun `http://people.brandeis.edu/~tbl/math20a/`

# *Previously on math 20a*

- We talked about the *cross product* of two vectors $\mathbf{a}$, $\mathbf{b}$, which is another vector
- We gave visual description of the cross product:
  - $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$
  - $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\phi$
  - $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ form a right handed set
- And we saw a method of computing the cross product:

$$\langle x, y, z\rangle \times \langle x', y', z'\rangle = \langle yz' - zy', zx' - xz', xy' - yx'\rangle$$

- Since these are the same, either of them could be taken as the definition, but the fact that they're the same is mysterious. Trust me that it works :).

# ***Examples: triple product calculations***

What is the volume of the parallelepiped with adjacent edges $PQ$, $PR$ and $PS$, where

$$P(2,0,-1), Q(4,1,0), R(3,-1,1), S(2,-2,2)$$

# *Examples: triple product calculations*

What is the volume of the parallelepiped with adjacent edges $PQ$, $PR$ and $PS$, where

$$P(2,0,-1), Q(4,1,0), R(3,-1,1), S(2,-2,2)$$

Well, we have

$$\mathbf{a} = \vec{PQ} = \langle 2,1,1\rangle, \mathbf{b} = \vec{PR} = \langle 1,-1,2\rangle, \mathbf{c} = \vec{PS} = \langle 0,-2,3\rangle$$

and we have that

$$\mathbf{a}\times\mathbf{b} = \langle (2)-(-1), 1-4, (-2)-(1)\rangle = \langle 3,-3,-3\rangle$$

and so $(\mathbf{a}\times\mathbf{b})\,\mathbf{.}\,\mathbf{c} = (3)0 + (-3)(-2) + (-3)(3) = 6 - 9 = -3$

# Examples: straight lines

What is the vector equation of the straight line through the points

$$P = (1, 2, 3) \quad Q = (1, 1, 1)$$

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Well, we have $\vec{PQ} = \langle 0, 1, 2 \rangle$, so the equation is

$$\mathbf{r}(t) = \langle 1, 1, 1 \rangle + \langle 0, 1, 2 \rangle t$$

# Examples: derivative of vector functions

Compute the derivative of each of the vector functions

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, e^t \rangle$$

$$\mathbf{r}_2(t) = \langle \frac{t}{t+1}, t^2 + t^3, e^t \sin t \rangle$$

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Well,

$$\mathbf{r}_1'(t) = \langle -\sin t, \cos t, e^t \rangle$$

And,

$$\mathbf{r}_2'(t) = \langle \frac{(t+1)(1) - (t)(1)}{(t+1)^2}, 2t + 3t^2, e^t \cos t + e^t \sin t \rangle$$

# ***Example: working backwards I***

A gun has muzzle speed 150$ms^{-1}$. We fire from ground level at angle $\theta$ to the horizontal. What is the shell's acceleration? Velocity? Position? When does it hit the ground? How far has it gone?

# ***Example: working backwards II***

A gun has muzzle speed 150$ms^{-1}$. We want to hit a target 800$m$ away. Give two angles of elevation which work.

## ***Examples: tangent vectors***

Compute the unit tangent vector to the function

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, e^t \rangle$$

at time $t = 2\pi$.

# Examples: tangent vectors

As before

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, e^t \rangle$$

gives $$\mathbf{r}_1'(t) = \langle -\sin t, \cos t, e^t \rangle$$

So $\mathbf{r}_1'(2\pi) = \langle 0, 1, e^{2\pi} \rangle$. To get the unit tangent vector, we just need to find a unit vector in the same direction. The length

$$|\mathbf{r}_1'(2\pi)| = \sqrt{1 + (e^{2\pi})^2}$$

and so the answer is

$$\frac{1}{|\mathbf{r}_1'(2\pi)|}\mathbf{r}_1'(2\pi) = \langle 0, \frac{1}{\sqrt{1 + (e^{2\pi})^2}}, \frac{e^{2\pi}}{\sqrt{1 + (e^{2\pi})^2}} \rangle$$

# Examples: arc length

Compute the arc length of the curve

$$\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \quad (0 \le t \le 1)$$

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We are meant to first work out

$$\mathbf{r}'(t) = 2t\mathbf{j} + 3t^2\mathbf{k} \quad (0 \le t \le 1)$$

Then do

$$\int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2} dt = \int_0^1 \sqrt{t^2(4 + 9t^2)} dt$$

$$= \int_0^1 t\sqrt{(4 + 9t^2)} dt = \int_4^{13} \frac{1}{18} \sqrt{u} du = \dots$$