Math 20A lecture 3

Lines and curves

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Announcements

- Homework due now, accepted until 10.45am
- A new homework will be posted on the class website this afternoon, due Sept 11th at the beginning of class
- Office hours are 3–4.30pm today
- See the website for all sorts of course-related fun
  http://people.brandeis.edu/~tbl/math20a/
Previously on math 20a

- We talked about the cross product of two vectors $a$, $b$, which is another vector.

- We gave visual description of the cross product:
  - $a \times b$ is perpendicular to $a$ and $b$
  - $|a \times b| = |a| |b| \sin \phi$
  - $a$, $b$ and $a \times b$ form a right handed set.

- And we saw a method of computing the cross product:

$$\langle x, y, z \rangle \times \langle x', y', z' \rangle = \langle yz' - zy', zx' - xz', xy' - yx' \rangle$$

- Since these are the same, either of them could be taken as the definition, but the fact that they’re the same is mysterious. Trust me that it works :).
Examples: triple product calculations

What is the volume of the parallelepiped with adjacent edges $PQ$, $PR$ and $PS$, where

\[ P(2, 0, -1), Q(4, 1, 0), R(3, -1, 1), S(2, -2, 2) \]
Examples: triple product calculations

What is the volume of the parallelepiped with adjacent edges $PQ$, $PR$ and $PS$, where

$$P(2, 0, -1), Q(4, 1, 0), R(3, -1, 1), S(2, -2, 2)$$

Well, we have

$$a = \vec{PQ} = \langle 2, 1, 1 \rangle, \ b = \vec{PR} = \langle 1, -1, 2 \rangle, \ c = \vec{PS} = \langle 0, -2, 3 \rangle$$

and we have that

$$a \times b = \langle (2) - (-1), 1 - 4, (-2) - (1) \rangle = \langle 3, -3, -3 \rangle$$

and so $(a \times b) \cdot c = (3)0 + (-3)(-2) + (-3)(3) = 6 - 9 = -3$
Examples: straight lines

What is the vector equation of the straight line through the points

\[ P = (1, 2, 3) \quad Q = (1, 1, 1) \]
Examples: straight lines

What is the vector equation of the straight line through the points

\[ P = (1, 2, 3) \quad Q = (1, 1, 1) \]

Well, we have \( \vec{PQ} = \langle 0, 1, 2 \rangle \), so the equation is

\[ \mathbf{r}(t) = \langle 1, 1, 1 \rangle + \langle 0, 1, 2 \rangle t \]
Examples: derivative of vector functions

Compute the derivative of each of the vector functions

\[ \mathbf{r}_1(t) = \langle \cos t, \sin t, e^t \rangle \]

\[ \mathbf{r}_2(t) = \langle \frac{t}{t+1}, t^2 + t^3, e^t \sin t \rangle \]
Examples: derivative of vector functions

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Well,

\[ \mathbf{r}'_1(t) = \langle -\sin t, \cos t, e^t \rangle \]
Examples: derivative of vector functions

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Well,

\[
\mathbf{r}'_1(t) = \langle -\sin t, \cos t, e^t \rangle
\]

And,

\[
\mathbf{r}'_2(t) = \langle \frac{(t + 1)(1) - (t)(1)}{(t + 1)^2}, 2t + 3t^2, e^t \cos t + e^t \sin t \rangle
\]
Example: working backwards 1

A gun has muzzle speed $150 \text{ms}^{-1}$. We fire from ground level at angle $\theta$ to the horizontal. What is the shell’s acceleration? Velocity? Position? When does it hit the ground? How far has it gone?
Example: working backwards II

A gun has muzzle speed $150 \, ms^{-1}$. We want to hit a target 800 m away. Give two angles of elevation which work.
Examples: tangent vectors

Compute the unit tangent vector to the function

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, e^t \rangle$$

at time $t = 2\pi$. 
Examples: tangent vectors

As before

\[ \mathbf{r}_1(t) = \langle \cos t, \sin t, e^t \rangle \]

gives

\[ \mathbf{r}'_1(t) = \langle -\sin t, \cos t, e^t \rangle \]

So \( \mathbf{r}'_1(2\pi) = \langle 0, 1, e^{2\pi} \rangle \). To get the unit tangent vector, we just need to find a unit vector in the same direction. The length

\[ |\mathbf{r}'_1(2\pi)| = \sqrt{1 + (e^{2\pi})^2} \]

and so the answer is

\[ \frac{1}{|\mathbf{r}'_1(2\pi)|} \mathbf{r}'_1(2\pi) = \langle 0, \frac{1}{\sqrt{1 + (e^{2\pi})^2}}, \frac{e^{2\pi}}{\sqrt{1 + (e^{2\pi})^2}} \rangle \]
Examples: arc length

Compute the arc length of the curve

\[ r(t) = i + t^2 j + t^3 k \quad (0 \leq t \leq 1) \]
Examples: arc length

Compute the arc length of the curve

\[ \mathbf{r}(t) = \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k} \quad (0 \leq t \leq 1) \]

We are meant to first work out

\[ \mathbf{r}'(t) = 2t \mathbf{j} + 3t^2 \mathbf{k} \quad (0 \leq t \leq 1) \]

Then do

\[
\int_0^1 |\mathbf{r}'(t)| \, dt = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2} \, dt = \int_0^1 \sqrt{t^2(4 + 9t^2)} \, dt \\
= \int_0^1 t \sqrt{4 + 9t^2} \, dt = \int_4^{13} \frac{1}{18} \sqrt{u} \, du = \ldots
\]