Math 20A lecture 4

Scratching surfaces

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Announcements

- Homework due this Friday, accepted until 10.45am
- Office hours are 2–3.30pm today
- See the website for all sorts of course-related fun
`http://people.brandeis.edu/~tbl/math20a/`

Previously on math 20a

- We talked about the *scalar triple product* of vectors, and had fun with Peter the pedagogical paper parallelepiped
- We followed ants on rollercoasters, seeing that a function
$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$
describes a path in space and time.
- I said (without evidence) that the best way of describing paths in 3D space (*without* a time element) is to *invent* a time element, tracing the path out in time.

Previously on math 20a

- We saw that when you have a path traced out in space and time, you can *differentiate* the components with respect to time t

$$\text{e.g. if } \mathbf{r}(t) = \langle t\sin(\theta), \cos(\theta t)\rangle$$

where θ is a constant, then

$$\mathbf{r}'(t) = \langle \sin(\theta), -\theta\sin(\theta t)\rangle.$$

- If we started with positions, this gives *velocity*; if we do it again, we get acceleration.

Previously on math 20a

- If we're dealing with paths in space, so the speed with which the path is traced out in time is not meaningful (arbitrary), we have to be careful
- The velocity vector, for instance, is not meaningful—it will change in magnitude based on the speed we traverse the path.
- But the *direction* of the velocity vector *is* meaningful (depends only on the curve).
- Thus, $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, a *unit* vector in the direction of the velocity vector, *is* meaningful. We call this the *unit tangent vector*.

Example: computing curvature

- Compute the curvature of the curve:

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

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We work step by step:

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 12t^2 - 6t^2, -6t, 2 \rangle = \langle 6t^2, -6t, 2 \rangle$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

Example: computing curvature cont

- Computing the curvature of the curve:

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

So far, we have:

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We then use the formula:

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

Example: computing curvature II

- Compute the curvature of the curve:

$$\mathbf{r}(t) = \langle \frac{3}{2}t^2, 2t^2 + t, \frac{2\sqrt{2}}{3}t^{3/2} \rangle$$

Example: computing curvature II

- Compute the curvature of the curve:

$$\mathbf{r}(t) = \langle \frac{3}{2}t^2, 2t^2 + t, \frac{2\sqrt{2}}{3}t^{3/2} \rangle$$

We work step by step:

$$\mathbf{r}'(t) = \langle 3t, 4t+1, \sqrt{2}t^{1/2} \rangle$$

$$\mathbf{r}''(t) = \langle 3, 4, (\sqrt{2}/2)t^{-1/2} \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(3t)^2 + (4t+1)^2 + (\sqrt{2}t^{1/2})^2}$$

$$= \sqrt{9t^2 + 16t^2 + 8t + 1 + 2t}$$

$$= \sqrt{25t^2 + 10t + 1} = \sqrt{(5t+1)^2} = 5t + 1$$

Example: computing curvature II cont

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- We continue:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle \frac{\sqrt{2t}}{2t}(-4t+1), \frac{\sqrt{2t}}{2t}3t, -3\rangle$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{\frac{1}{2t}(1 - 8t + 16t^2 + 9t^2 + 18t)}$$
$$= \sqrt{\frac{1}{2t}(1 + 10t + 25t^2)}$$

Example: tangents, normals, binormals (oh my!)

- Compute $\mathbf{T}$,$\mathbf{N}$,$\mathbf{B}$ for the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$

Example: tangents, normals, binormals (oh my!)

- Compute $\mathbf{T}$,$\mathbf{N}$,$\mathbf{B}$ for the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$
  We work step by step:

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\cos t, -\sin t, 0 \rangle$$

$$\mathbf{B}(t) = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$$

Example: computing curvature II cont

- Computing the curvature of the curve:

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

So far, we have:

$$|\mathbf{r}'(t)| = 5t + 1$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{\frac{1}{2t}(1 + 10t + 25t^2)} = \frac{1}{\sqrt{2t}}(5t + 1)$$

Example: computing curvature II cont

- Computing the curvature of the curve:

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We then use the formula:

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{(5t+1)/\sqrt{2t}}{(5t+1)^3} = \frac{1}{\sqrt{2t}(5t+1)^2}$$

Matching game!

- Match the pictures you see to the following parametric curves:

$$\mathbf{r}_1(u,v) = \langle (\sin v + 1.5)\sin u, (\sin v + 1.5)\cos u, v\rangle$$

$$\mathbf{r}_2(u,v) = \langle \sin u, v, \cos u + \sin v\rangle$$

$$\mathbf{r}_3(u,v) = \langle u, v, \sin(2u - 2v)\rangle$$

$$\mathbf{r}_4(u,v) = \langle 0.1(3 + v + 2v^3)\sin u, 0.1(3 + v + 2v^3)\cos u, v\rangle$$

$$\mathbf{r}_5(u,v) = \langle u, v, \frac{3v}{u^2 + v^2 + 1}\rangle$$

$$\mathbf{r}_6(u,v) = \langle \sin u + 1.5\sin v, \cos u + 1.5\sin v, v\rangle$$

- There is an odd one out!