Math 20A lecture 9

Tangent planes

T.J. Barnet-Lamb

tbl@brandeis.edu

Brandeis University

Announcements

- Homework four due today, accepted until 10.50am
- Homework five posted, due a week from today
- Office hours are 3–4.30pm today
- See the website for all sorts of course-related fun `http://people.brandeis.edu/~tbl/math20a/`
- Grades should be posted on LATTE. It is *your responsibility* to log into LATTE and check that the grades are entered correctly. You have one week from the day each homework score is posted to alert me of any inaccuracies.

Previously on math 20a

- We talked about the chain rule.
- If we have a function F of two *intermediate* variables (let's say there are two, x and y), each of which depends on a *final* variable (t say), then
- The chain rule tells us that

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

Chain rule example

- We have a function $F(x, y)$ of two variables, and we know:

$$F_x(0,0) = 3 \quad F_y(0,0) = 4 \quad F_x(1,1) = 1 \quad F_y(1,1) = 5$$

We let $g(t) = F(t^2, e^t)$. What is $f'(1)$?

Chain rule example

- We have a function $F(x, y)$ of two variables, and we know:

$$F_x(0,0) = 3 \quad F_y(0,0) = 4 \quad F_x(1,1) = 1 \quad F_y(1,1) = 5$$

  We let $g(t) = F(t^2, e^t)$. What is $f'(1)$?

- The chain rule tells us that

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{d(t^2)}{dt} + \frac{\partial g}{\partial y}\frac{d(e^t)}{dt}$$

- We get

$$\frac{dg}{dt} = F_x(x, y)2t + F_y(x, y)e^t$$

Tangent plane example

- Find the tangent plane to the function

$$z = y\cos(x - y)$$

at the point $(2, 2, 2)$.

Tangent plane example

- Find the tangent plane to the function

$$z = y\cos(x-y)$$

at the point $(2,2,2)$.

- We first work out the partial derivatives

$$\frac{\partial z}{\partial x} = -y\sin(x-y) \quad \frac{\partial z}{\partial y} = \cos(x-y) + y\sin(x-y)$$

- and plug in the values $x=2, y=2$:

$$\frac{\partial z}{\partial x} = (-2)(0) = 0 \quad \frac{\partial z}{\partial y} = \cos(0) + 2\sin(0) = 1$$

Tangent plane example

- Find the tangent plane to the function

$$z = y\cos(x-y)$$

  at the point $(2, 2, 2)$.

- and plug in the values $x = 2, y = 2$:

$$\frac{\partial z}{\partial x} = (-2)(0) = 0 \quad \frac{\partial z}{\partial y} = \cos(0) + 2\sin(0) = 1$$

- Then we use the formula $z = (x-x_0)\frac{\partial z}{\partial x} + (y-y_0)\frac{\partial z}{\partial y} + z_0$, which gives $z = (x-2)0 + (y-2)1 + 2$ or $z = y$

Linear approximation example

- Approximate the value of $(4.02)^{1.99}$, without using a calculator. (You may use the fact that $\ln 4 = 1.38629$.)

Linear approximation example

- Approximate the value of $(4.02)^{1.99}$, without using a calculator. (You may use the fact that $\ln 4 = 1.38629$.)
- We are asked to work out an approximation to the value of the function $f(x, y) = x^y$ where $x = 4.02$, $y = 1.99$. We will approximate this value using the tangent plane at $(x_0, y_0) = (4, 2)$.
- The partials are $f_x = x^{y-1}y$, $f_y = (\ln x)x^y$. At $(4, 2)$ these are $f_x = 8, f_y = 1.38629 \times 16 = 22.18$
- Use the formula $z = (x - x_0)\frac{\partial z}{\partial x} + (y - y_0)\frac{\partial z}{\partial y} + z_0$; we get $z = (x - 4)8 + (y - 2)22.18 + 16$
- Plugging in $x = 4.02, y = 1.99$, we get $z = 0.02 * 8 - 0.01 * 22.18 + 16 = 15.94$

Linear approximation example II

- A model for the surface area (in sq ft) of a human body is given by

$$S = 0.1091 w^{0.425} h^{0.725}$$

where w s the weight in pounds, and h is the height in inches. Fred weighs 120 pounds and is 72 inches tall. Work out his surface area using the formula. If there is a ± 2% error in each measurement, what is the (approximate) resulting error in the surface area.

Linear approximation example II

- A model for the surface area (in sq ft) of a human body is given by
$$S = 0.1091 w^{0.425} h^{0.725}$$
where w s the weight in pounds, and h is the height in inches. Fred weighs 120 pounds and is 72 inches tall. Work out his surface area using the formula. If there is a ± 2% error in each measurement, what is the (approximate) resulting error in the surface area.
- Plugging in, $S = 0.1091(120)^{0.425}(72)^{0.725} = 18.5371$
- We approximate the error using the tangent plane. We work out the partials

$$S_w = (0.1091)(0.425) w^{-0.575} h^{0.725}, S_h = (0.1091) 0.725 w^{0.425} h^{-0.275}$$

Linear approximation e.g. ctd.

- We approximate the error using the tangent plane. We work out the partials

$$S_w = (0.1091)(0.425)w^{-0.575}h^{0.725}, S_h = (0.1091)0.725w^{0.425}h^{-0.275}$$

- We then plug in

$$S_w = 0.0656524 \text{ sq ft/lb}, S_h = 0.186659 \text{ sq ft/in}$$

- The error in w could be up to 2% of 120, or 2.4 lb. This would lead to an $0.0656524 \times 2.4 = 0.157566$ sq ft error in S.
- The error in h could be up to 2% of 72, or 1.44 inches. This would lead to an $0.186659 \times 1.44 = 0.268789$ sq ft error in S.

Linear approximation e.g. ctd.

- The error in w could be up to 2% of 120, or 2.4 lb. This would lead to an $0.0656524 \times 2.4 = 0.157566$ sq ft error in S.
- The error in h could be up to 2% of 72, or 1.44 inches. This would lead to an $0.186659 \times 1.44 = 0.268789$ sq ft error in S.
- Worst possible total error (approx) is $0.157566 + 0.268789 = 0.426355$ sq ft
- Worst possible error is ± 0.426355 sq ft

Tangent planes to parametric surfaces

- Find an equation to the tangent plane of the parametric surface
$$\mathbf{r}(u, v) = \langle uv, u \sin v, v \cos u \rangle$$
at the point corresponding to $u = 0, v = \pi$.

Tangent planes to parametric surfaces

- Find an equation to the tangent plane of the parametric surface

$$\mathbf{r}(u, v) = \langle uv, u \sin v, v \cos u \rangle$$

at the point corresponding to $u = 0, v = \pi$.

- First we work out which point corresponds to $u = 0, v = \pi$ by plugging in

$$\mathbf{r}(0, \pi) = \langle 0, 0, \pi \rangle$$

thus we know the tangent plane goes through this point.

Tangent planes to parametric surfaces

- Find an equation to the tangent plane of the parametric surface

$$\mathbf{r}(u,v) = \langle uv, u\sin v, v\cos u\rangle$$

at the point corresponding to $u = 0, v = \pi$.

- Now we consider u fixed at 0, and differentiate w.r.t. v to get the tangent vector to the curve $\mathbf{r}(t) = \mathbf{r}(0,t)$

$$\mathbf{r}_v = \langle 0, 0, 1\rangle$$

Plugging in $v = \pi$, we get $\mathbf{T} = \langle 0, 0, 1\rangle$. *The tangent plane is parallel to this.*

Tangent planes to parametric surfaces

- Find an equation to the tangent plane of the parametric surface
$$\mathbf{r}(u,v) = \langle uv, u\sin v, v\cos u\rangle$$
at the point corresponding to $u = 0, v = \pi$.
- Now we consider v fixed at π, and differentiate w.r.t. u to get the tangent vector to the curve $\mathbf{r}(t) = \mathbf{r}(t, 0)$
$$\mathbf{r}_u = \langle \pi, 0, -\pi\sin u\rangle$$
Plugging in $u = 0$, we get $\mathbf{T} = \langle \pi, 0, 0\rangle$. *The tangent plane is parallel to this.*

Tangent planes to parametric surfaces

- We want to find a plane through $(0,0,\pi)$, parallel to $\langle 0,0,1\rangle$ and $\langle \pi,0,0\rangle$.
- The normal to the plane will be $\langle \pi,0,0\rangle \times \langle 0,0,1\rangle$; that is, $\langle 0,-\pi,0\rangle$. Thus we know the plane takes the form $\mathbf{n}\cdot\mathbf{r} = c$, or $-\pi y = c$. We just need to figure out c.
- We do this using the fact that $(0,0,\pi)$ is on the plane. We get $c = 0$. Thus the plane is $-\pi y = 0$.