7. We then asked the following question: what collections of ‘pieces’ (elements of $\Diamond$) would have the property that we could build everything else out of them using just field elements and addition. We saw that there seemed to be several collections that would ‘do the job’, and none seemed ‘better’ than the others. We had $t - 1, t^2 - 1, t^3 - 1$, we had $(t - 1), (t - 1)t, (t - 1)t^2$ and we had $(t - 1), (t - 1)^2, (t - 1)^3$.

We noticed that while these collections were quite varied, they all had the same number of elements. Our main goal for this section of the course will be to prove that things always work out this way.

Before we do that, let’s just write out formally what we did informally last time.

**Definition 1.** Let $F$ be a field. A vector space over $F$ is a set $V$ equipped with two operations: one (which we write $+$) takes two elements of $V$ and gives us another such element, and the other (which we write $\cdot$) takes an element of $V$ and an element of $F$ and gives us back an element of $V$. (Technically, $+$ is just a function from $V \times V$ to $V$ and $\cdot$ is a function from $F \times V$ to $V$.) These operations must satisfy the following properties:

A1 There is some element $0 \in V$ with the property that $0 + v = v$ for all $v \in V$.

A2 $(u + v) + w = u + (v + w)$ for all elements $u, v, w \in V$.

A3 $u + v = v + u$ for all elements $u, v \in V$.

A4 For every element $v \in V$, there is an element $w$ such that $v + w = 0$.

M1 For every pair of elements $a, b \in F$, and $v \in V$, we have that $(ab) \cdot v = a \cdot (b \cdot v)$.

D1 For every pair of elements $a, b \in F$, and $v \in V$, we have that $(a + b) \cdot v = (a \cdot v) + (b \cdot v)$

D2 For elements $a \in F$, and pair of elements $v, w$ in $V$, we have that $a \cdot (v + w) = (a \cdot v) + (a \cdot w)$

As with fields before, we at first need to be careful and fully bracket everything so we only ever combine two things together at a time (since that’s all the rules allow). On the other hand, one we internalize associativity, and axiom M1 we can lighten up a little bit, and write things like $u + v + w$ and $abv$, since we know it doesn’t make any difference which way we work them out. We will similarly make the usual convention that in an expression like $a \cdot v + a \cdot w$, we always ‘do the multiplications first’, even if we don’t bracket things out.

We can also do a few sanity checks:

**Proposition 2.** In all the following, suppose that $V$ is a vector space over a field $F$.

1. Suppose $v \in V$. Then there is at only one element $w$ with $w + v = 0$. (We’re thus entitled to give it a name, $-(v)$.)

2. For all $v \in V$, $0v = 0$.

3. For all $v \in V$ and $a \in F$, $-a \cdot v = -(a \cdot v)$.

4. If $v$ is a vector, $v \neq 0$, and $a$ is a scalar such that $a \cdot v = 0$, then $a = 0$.

**Proof.** Exercise.

**Definition 3.** A set $S$ of elements in a vector space $V$ is a spanning set for $V$ if and only if, whenever $v \in V$, we can find a finite subcollection $\{s_1, \ldots, s_k\}$ of elements of $S$, and elements $\alpha_1, \ldots, \alpha_k \in F$ such that $v = \alpha_1s_1 + \cdots + \alpha ks_k$. (In this case we say that we can express $v$ as a linear combination of the elements $s_1, \ldots, s_k$.)
Definition 4. A spanning set $S$ for a vector space $V$ contains redundancy if and only if there is some $v \in S$ such that $S \setminus \{v\}$ is still a spanning set for $V$. ($S \setminus \{v\}$ means all the collection containing all the elements of $S$ except $v$.)

Definition 5. A basis for a vector space is a spanning set which contains no redundancy.

It will be helpful to devise a notion of when a collection of vectors ‘contains no wasted space’ which applies even when the collection is not a spanning set.

Definition 6. A set $S$ of elements in a vector space $V$ is linearly independent if and only if, whenever have a finite subcollection $\{s_1, \ldots, s_k\}$ of elements of $S$, and elements $\alpha_1, \ldots, \alpha_k \in F$ such that $0 = \alpha_1 s_1 + \cdots + \alpha_k s_k$, it must be that all the $\alpha_i = 0$.

Proposition 7. Suppose $S$ is a spanning set in a vector space. The following are equivalent:

1. $S$ contains no redundancy.
2. $S$ is linearly independent