Proposition 7. Suppose $S$ is a spanning set in a vector space. The following are equivalent:

1. $S$ contains no redundancy.
2. $S$ is linearly independent

Proof. First, let us show that if the vectors are linearly independent, then there is no redundancy. That is the same thing as showing that if there is redundancy, then the vectors are linearly dependent. So let us assume that there is redundancy. Then there is some vector $v \in S$ such that $S \setminus \{v\}$ is still a spanning set. That means, in turn that we can find some $v_1', \ldots, v_k' \in S \setminus \{v\}$ (all different) and coefficients $\alpha_1, \ldots, \alpha_k$ such that

$$v = \alpha_1 v_1' + \alpha_2 v_2' + \cdots + \alpha_k v_k'$$

Then

$$0 = \alpha_1 v_1' + \alpha_2 v_2' + \cdots + \alpha_k v_k' + (-1)v$$

showing that the vectors of $S$ are linearly dependent.

Conversely, let us show that if the vectors are linearly dependent, then there is redundancy. By definition of ‘linearly dependent’, we can find $v_1, \ldots, v_k$ and coefficients $\alpha_1, \ldots, \alpha_k$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

We said that $\alpha_1, \ldots, \alpha_k$ are not all zero, so at least one of them is nonzero. Reordering, we can suppose that it’s the first one. Then we have

$$v_1 = \alpha_2(-1/\alpha_1)v_2 + \alpha_3(-1/\alpha_1)v_3 + \cdots + \alpha_k(-1/\alpha_1)v_k$$

Now, I claim that there is redundancy in that $S \setminus \{v_1\}$ is still a spanning set. To see this, we must show that given any vector $v$, we can make $v'$ out of the vectors of $S \setminus \{v_1\}$. So let’s suppose we’re given a $v$. Since $S$ was a spanning set to start with, we can certainly find $s_1, \ldots, s_k$ in $S$ and coefficients $\lambda_1, \ldots, \lambda_k$ such that:

$$v = \lambda_1 s_1 + \cdots + \lambda_k s_k$$

Now, if none of the $s_i$ is $v_1$, we’ve already made $v$ out of vectors from $S \setminus \{v_1\}$, and we are done. Otherwise, one of the $s_i$ (reordering, let’s say $s_1$) is $v_1$.

$$v = \lambda_1 v_1 + \lambda_2 s_2 + \cdots + \lambda_k s_k$$

expressing $v$ as a linear combination of the $v_i$ and $s_i$; i.e., as a linear combination of elements of $S \setminus \{v_1\}$.

In fact, the method of our proof gave us the following fact, which we can record for future reference

Proposition 8. If $S$ is a spanning set and $v \in S$ can be expressed as a linear combination of the other elements of $S$, then $S \setminus \{v\}$ is a spanning set.

Lemma 9. Suppose $S = \{v_1, \ldots, v_k\}$ is a finite spanning set for a vector space $V$, and $v \in V$. Then we can write $v = \alpha_1 v_1 + \cdots + \alpha_k v_k$ for some coefficients $\alpha_i$.

$^3$Technically, it’s possible that several of the $s_i$ are $v_1$ in disguise. But in this case we can just combine them into one single multiple of $v_1$. 
Lemma 10. Any subset of a linearly independent set is linearly independent.

Proof. Suppose that $S \subset T$. We want to show that if $T$ is linearly independent then $S$ is; equivalently, we will show that if $S$ is linearly dependent then $T$ is. So assume $S$ linearly dependent, so we can write

$$0 = \alpha_1 s_1 + \cdots + \alpha_k s_k$$

where the $s_i$ are in $S$ and the $\alpha_i$ are elements of the field and not all zero. Then, since the $s_i$ are also elements of $T$, this shows $T$ is linearly dependent. \qed

Lemma 11. Any superset of a spanning set is a spanning set.