The point of introducing linear independence is the following key lemma:

**Lemma 12** (The Steinitz exchange lemma). We work with a vector space $V$. Suppose that $A$ is a finite spanning set, and $B$ is a finite linearly independent set. Write $\#B$ for the number of elements of $B$. Then we can remove $\#B$ elements from $A$, getting a smaller set $A'$ such that $A' \cup B$ is still a spanning set.

In particular, this means that $A$ must contain at least $\#B$ elements.

**Proof.** We proceed by induction on the number of elements of the set $B$. If the set $B$ is empty, then what we are asked to prove is trivial. So let’s assume that $B$ contains at least one element. Then we can single out one element from $B$ from the rest, writing $B = \{b_1\} \cup \{b_2,\ldots,b_k\}$. (So $k = \#B$.)

Now, $\{b_2,\ldots,b_k\}$ is still a linearly independent set, being a subset of $B$, and it has $k - 1$ elements: fewer elements than $B$. Thus by induction, we may assume that our result is true for $A$ and $\{b_2,\ldots,b_k\}$, and we can eliminate $k - 1$ elements from $A$, getting a new set $A''$ such that $A'' \cup \{b_2,\ldots,b_k\}$ is a spanning set. We’ll write $B_1 = \{b_2,\ldots,b_k\}$ for short, and we’ll spell out the elements of $A''$ as $\{a_2'',\ldots,a_r''\}$.

Since $A'' \cup B_1$ is a spanning set, we can write $b_1$ as a linear combination of the elements of $A'' \cup B_1$:

$$b_1 = \alpha_1 a_1'' + \alpha_2 a_2'' + \cdots + \alpha_r a_r'' + \beta_2 b_2 + \cdots + \beta_k b_k$$

Now, we can’t have all of the $\alpha_i$ be zero, otherwise

$$0 = (-1)b_1 + \beta_2 b_2 + \cdots + \beta_k b_k$$

which would mean that the $b_i$ were linearly dependent, contrary to assumption. Thus some $\alpha_i$ is nonzero; reordering, let’s suppose it’s the first one. Then

$$a_1'' = (1/\alpha_1)b_1 + (-1/\alpha_1)\alpha_2 a_2'' + \cdots + (-1/\alpha_1)\alpha_r a_r'' + (-1/\alpha_1)\beta_2 b_2 + \cdots + (-1/\alpha_1)\beta_k b_k$$

Now, since $A'' \cup \{b_2,\ldots,b_k\}$ is a spanning set, so $A'' \cup B$ is a spanning set; and thus, since $a_1''$ can be expressed as a linear combination of other elements of $A'' \cup B$, $\{a_2'',\ldots,a_r''\} \cup B$ is a spanning set. But then we are done, taking $A' = \{a_2'',\ldots,a_r''\}$. \qed

**Corollary 13.** Suppose $V$ is a vector space, and $B_1$ and $B_2$ are two finite bases for $V$. Then $\#B_1 = \#B_2$.

**Proof.** Since $B_1$ is a finite spanning set and $B_2$ is a finite linearly independent set, the lemma tells us $B_2$ has at least as many elements as $B_1$. On the other hand, since $B_2$ is a finite spanning set and $B_1$ is a finite linearly independent set, the lemma tells us $B_1$ has at least as many elements as $B_2$. We are therefore done. \qed

**Corollary 14.** Suppose $V$ is a vector space, and $B_1$ and $B_2$ are two bases, one of which is infinite. Then the other is also infinite.

**Proof.** Suppose for contradiction that $B_1$ is infinite, and $B_2$ is finite. Let $m$ denote the number of elements of $B_2$. Since $B_1$ is infinite, we can find some $m + 1$ elements of $B_1$, making a finite set $B_1'$ which is linearly independent, being a subset of $B_1$, which was a basis and hence linearly independent. Since $B_1'$ is linearly independent, and $B_2$ is a spanning set, $B_1'$ must have no more elements than $B_2$, so $m + 1 \leq m$, a contradiction. \qed

**Corollary 15.** Suppose $V$ is a vector space. Any two bases for $V$ have the same number of elements, in the sense that either they are both infinite, or they are both finite with the same number of elements.

**Fact 16.** Every vector space has some basis.