§5. Field extensions—We now return to the study of fields and rings, and using two new tools we have acquired in the interim (vector spaces and unique factorization of polynomials). The focus of our study will move from individual fields to fields in combination. In particular, we will study a situation which we call a field extension; two fields, $L$ and $K$, where $K$ is a subfield of $L$. We call this ‘a field extension $L$ over $K$’, and write ‘a field extension $L/K$’.

The reason we are interested in this situation is that while (as we saw) fields are great for solving several different kinds of equation (e.g. linear equations, simultaneous linear equations), they can’t necessarily solve all equations—not even quadratic equations. But often we can find another field in which the equations do have solutions. Comparing these two fields will then give us a great deal of information about ‘how unsolvable’ the equation was before.

For instance, the equation $x^2 = 2$ has no solution over $\mathbb{Q}$, but does have solutions over $\mathbb{C}$, or over $\{a + \sqrt{2}b|a, b \in \mathbb{Q}\}$. (In fact, every equation has a solution over $\mathbb{C}$; this is the ‘fundamental theorem of algebra’.)

The first thing we must do is divide the elements of $L$ into two different kinds.

**Definition 1.** Suppose $L/K$ is a field extension, and $\alpha \in L$. We say that $\alpha$ is algebraic over $K$ if there is some polynomial $p(t) \in K[T]$ such that $p(\alpha) = 0$. We say that $\alpha$ is transcendental over $K$ otherwise. (Sometimes we leave out the ‘over $K$’, when it is obvious from the context which $K$ we have in mind.)

**Example 2.** Let us consider the field extension $\mathbb{R}/\mathbb{Q}$. It is easy to give examples of algebraic elements in $\mathbb{R}$: $\sqrt{2}$, $\sqrt{6}$, $\sqrt{2} + 1$, $\sqrt{2} + \sqrt{3}$. It is much harder to come up with transcendental elements. The first example given was the Liouville number, $\sum_k 10^{-k!}$. (One of the first really weird things that happened in the creation of formal set theory was Cantor’s much easier proof which shows that in fact ‘most’ numbers are transcendental—without giving a single example of one!)

Later in the nineteenth century, it was shown that $e$ and $\pi$ are transcendental.