Throughout this lecture, $K$ will be a subfield of $\mathbb{R}$.

**Lemma 7.** Suppose that $\Gamma$ is a circle defined over $K$. Suppose that $\ell$ is a line defined over $K$. Suppose finally that $\Gamma$ and $\ell$ intersect in at least one point, $P$ say. Then we can find a subfield $K'$ of $\mathbb{R}$ containing $K$, such that:

1. $P$ is defined over $K$, and

**Proof.** By the assumption that $\Gamma$ and $\ell$ are defined over $K$, we can write equations:

$$x^2 + y^2 + Ax + By + C = 0$$

and

$$ax + by + c = 0$$

for $\Gamma$ and $\ell$ respectively, where $A$, $B$, $C$, $a$, $b$ and $c$ are in $K$.

Now, we know that one of $a$ and $b$ is not 0. We will handle the case where $a \neq 0$; if $b \neq 0$, we could proceed with exactly the same argument with the roles of $x$ and $y$ swapped over. (This is an example of making an assumption without loss of generality.) Then given that $a \neq 0$, we can divide the line equation through by $a$ rewriting it as

$$x + by + c = 0$$

(where this $b$ and $c$ might be different to the original ones).

Let $(\chi, \gamma)$ be the coordinate of $P$, and let $K' = K[\gamma] \subset \mathbb{R}$. We will show that this satisfies the two properties we require. We first see that $P$ is defined over $K'$. Obviously, $\gamma \in K'$, and it is also the case that $\chi = -b\gamma - c$ is in $K'$. Thus $P$ is defined over $K'$. It remains to show $[K' : K] = 2$

We know that $(\chi, \gamma)$ satisfies the equation of $\Gamma$, so

$$\chi^2 + \gamma^2 + A\chi + B\gamma + C = 0$$

and then, using the fact that $\chi = -b\gamma - c$, we see that

$$(b^2 + 1)\gamma^2 + 2bc\gamma + c^2 + B\gamma - Ab\gamma - Ac + C = 0$$

that is,

$$(b^2 + 1)\gamma^2 + (2bc + B - Ab)\gamma + c^2 + C = 0$$

So $\gamma$ satisfies the polynomial

$$P(T) = (b^2 + 1)T^2 + (2bc + B - Ab)T + c^2 + C = 0.$$  

The coefficients of this polynomial are in $K$. Thus the minimal polynomial of $\gamma$ divides into $P(T)$, and so the degree of the minimal polynomial is at most the degree of $P(T)$. So the degree of the minimal polynomial is 1 or 2.

But then $[K[\gamma] \subset \mathbb{R} : K] = 2$ or $[K[\gamma] \subset \mathbb{R} : K] = 1$. $\square$

**Lemma 8.** Suppose $\Gamma_1$ and $\Gamma_2$ are two circles which are defined over $K$. Suppose also that $\Gamma_1$ and $\Gamma_2$ intersect in at least one point, $P$ say. Then we can find a subfield $K'$ of $\mathbb{R}$ containing $K$, such that:

1. $P$ is defined over $K$, and


Proof. By the assumption that $\Gamma_1$ and $\Gamma_2$ are defined over $K$, we can write equations:

\[ x^2 + y^2 + A_1 x + B_1 y + C_1 = 0 \]

and

\[ x^2 + y^2 + A_2 x + B_2 y + C_2 = 0 \]

for $\Gamma_1$ and $\Gamma_2$ respectively, where $A, B, C, a, b$ and $c$ are in $K$. We know that $P$ is a solution to these two equations. But that means it is a solution both to the difference of those two equations, viz:

\[(A_1 - A_2)x + (B_1 - B_2)y + (C_1 - C_2) = 0\]

which describes a line, $\ell$ say, defined over $K$. Thus $P$ is on the intersection of $\Gamma_1$ and $\ell$, and hence (by the previous lemma) we can find a subfield $K'$ of $\mathbb{R}$ containing $K$ with the properties we require. \qed