## Preface

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Preface:

1. If the world does not obey a model’s predictions, we can decide that the model needs improvement or that the world is wrong, that some assets are “mispriced” and present trading opportunities for the shrewd investor.
2. Asset pricing theory all stems from one simple concept: price equals expected discounted payoff.
3. Absolute pricing: we price each asset by reference to its exposure to fundamental sources of macroeconomic risk.
4. Relative pricing: we ask what we can learn about an asset’s value given the prices of some other assets. We don’t ask where the prices of the other assets came from, and we use as little information about fundamental risk factors as possible.
5. Standard macroeconomic models predict that people really do not care much about business cycles (Lucas[1987]). Asset prices reveal that they do – that they forego substantial return premia to avoid assets that fall in recessions. This fact ought to tell us something about recessions.

\[ p_t = E(m_{t+1}x_{t+1}) \]

6. \[ m_{t+1} = f(data, parameters) \]
   where \( p = \) asset price, \( x_{t+1} = \) asset payoff, \( m = \) stochastic discount factor.
7. The discount factor approach is also associated with a state-space geometry in place of the usual mean-variance geometry, and this book emphasizes the state-space intuition behind many classic results.

Chapter 1  Consumption-Based Model and Overview

1. Some basics
   1. An investor must decide how much to save and how much to consume. The marginal utility loss of consuming a little less today and buying a little more of the asset should equal the marginal utility of consuming a little more of the asset’s payoff in the future. It follows that the asset’s price should equal the expected discounted value of the asset’s payoff, using the investor’s marginal utility to discount the payoff.
   2. Interest rates are related to expected marginal utility growth, and hence to the expected path of consumption. Therefore, high real interest rates should be associated with an expectation of growing consumption.
   3. Marginal utility, not consumption, is the fundamental measure of how you feel. Consumption is low when marginal utility is high. Consumption is also low and marginal utility is high when the investor’s other assets have done poorly; thus we may expect that prices are low for assets that covary positively with a large index such as the market portfolio.
4. Basic consumption-based model:

\[ p_t = E_t\left[ \beta u'(c_{t+1}) x_{t+1} \right], \text{ where } p_t u'(c_t) \text{ is the loss in utility if the investor buys another unit of the asset; } E_t(\beta u'(c_{t+1}) x_{t+1}) \text{ is the increase in (discounted, expected) utility he obtains from the extra payoff at } t+1. \text{ The investor continues to buy or sell the asset until the marginal loss equals the marginal gain. Beta captures impatience, and is called the subjective discount factor.} \]

5. \( p_t = E(m_{t+1} x_{t+1}) \), one can incorporate all risk corrections by defining a single stochastic discount factor (pricing kernel or state-price density) – the same one for each asset – and put it inside the expectation. \( M \) is stochastic or random because it is not known with certainty at time \( t \). The correlation between the random components of the common discount factor \( m \) and the asset-specific payoff \( x \) generate asset-specific risk corrections.

- All asset pricing models amount to alternative ways of connecting the stochastic discount factor to data.

6. Divide both sides of the equation \( p_t = E(m_{t+1} x_{t+1}) \) by \( p \), \( 1 = E(m_{t+1} R_{t+1}) \). If pricing the excess return (zero-cost portfolio), \( 0 = E[m_{t+1} (R_{t+1}^a - R_{t+1}^b)] \). Zero price does not imply zero payoff.

II. Classic issues in finance

1. Risk-free rate: \( R^f = 1/E(m) \), if a risk-free security is not traded, we can define this as the “shadow” risk-free rate (zero-beta rate).

Let assume power utility \( u'(c_t) = c^{-\gamma} \) and \( \ln\left(\frac{c_{t+1}}{c_t}\right) \sim N(\mu, \sigma^2) \rightarrow \)

\[ r_i^f = \delta + \gamma E_t[\ln\left(\frac{c_{t+1}}{c_t}\right)] - \frac{\gamma^2}{2} \sigma_t^2 \left[ \ln\left(\frac{c_{t+1}}{c_t}\right) \right] \]

**Real interest rates are high when people are impatient (delta),** when expected consumption growth is high (intertemporal substitution), or when risk is low (precautionary saving, sigma). A more curved utility function (\( \gamma \)) or a lower elasticity of intertemporal substitution (\( 1/\gamma \)) means that interest rates are more sensitive to changes in expected consumption growth.

2. Risk corrections:

\[ p_t = E(m) E(x) + \text{cov}(m, x) = \frac{E(x)}{R^f} + \text{cov}(m, x), \text{ where first term is the price in a risk-neutral world – if consumption is constant or if utility if linear. The second term is a risk adjustment.} \]

\( 1 = E(m) E(R^i) + \text{cov}(m, R^i) = \frac{E(R^i)}{R^f} + \text{cov}(m, R^i) \)

\( E(R^i) - R^f = -R^f \text{ cov}(m, R^i) \)
3. Only the component of a payoff perfectly correlated with the discount factor generates an extra return. Idiosyncratic risk, uncorrelated with the discount factor, generates no premium.

\[ \text{If } \text{cov}(x,m) = 0 \Rightarrow P_t = \frac{E(x)}{R^f}. \]

\[ X = \text{proj}(x|m) + e. \]

- Projection means linear regression without a constant

\[ \text{Proj}(x|m) = \frac{E(mx)}{E(m)} - m, \text{p(proj}(x|m)) = p(x). \]

4. Expected return-beta representation

\[ E(R^i) - R^f = -R^f \text{cov}(m, R^i) = -\frac{\text{cov}(m, R^i)}{E(m)} = \frac{\text{cov}(m, R^i)}{\text{var}(m)} \left[ -\frac{\text{var}(m)}{E(m)} \right] \]

\[ \Rightarrow E(R^i) = R^f + \beta_{i,m} \lambda_m, \lambda \text{ is the same for all assets } I, \text{while beta varies from asset to asset. The } \lambda \text{ is interpreted as the price of risk and the beta as the quantity of risk in each asset.} \]

5. Mean-Variance frontier: all asset returns lie inside a mean variance frontier. Assets on the frontier are perfectly correlated with each other and with the discount factor. Returns on the frontier can be generated as portfolios of any two frontier returns. We can construct a discount factor from any frontier return (except Rf), and an expected return-beta representation holds using any frontier return (except Rf) as the factor.

\[ 1 = E(m) E(R^i) + \text{cov}(m, R^i) = \frac{E(R^i)}{R^f} + \rho_{i,m} \sigma_i \sigma_m \]

\[ E(R^i) = R^f - \rho_{i,m} \frac{\sigma_m}{E(m)} \sigma_i \]

\[ \left| E(R^i) - R^f \right| \leq \frac{\sigma_m}{E(m)} \sigma_i \]

- Since each point in the mean variance frontier is perfectly correlated with the discount factor, we must be able to pick constants a, b, d, e such that

\[ m = a + b R^{\text{mv}} \]

\[ R^{\text{mv}} = d + e m \]

Thus, any mean variance efficient return carries all pricing information. Given a mean variance efficient return and the risk-free rate, we can find a discount factor that prices all assets and vice versa.

- Expected returns can be described in a single-beta representation using any mean variance efficient return (except for Rf)

\[ E(R^i) = R^f + \beta_{i,\text{mv}} \left[ E(R^{\text{mv}}) - R^f \right] \]
- The essence of the beta pricing model is that, even though the means and standard deviations of returns fill out the space inside the mean variance frontier, a graph of mean returns versus betas should yield a straight line.

6. Slope of the mean-standard deviation frontier and equity premium puzzle.

\[
\frac{E(R_i^f) - R_i^f}{\sigma_i} \leq \frac{\sigma_m}{E(m)} \approx \gamma \sigma (\Delta \ln c), \text{ The slope of the mean-standard deviation frontier is the largest available Sharpe ratio. It is higher if the economy is riskier – if consumption is more volatile – or if investors are more risk averse.}
\]

7. In most cases, the discount factor is a function of aggregate variables (market return, aggregate consumption), so it is plausible to hold the properties of the discount factor constant as we compare one individual asset to another.

8. In continuous time: \(0 = \Lambda dt + E_i[d(\Lambda p)]\) (analogue to \(p = E(mx)\)).

\[
E_i\left(\frac{dp}{p}\right) + \frac{D}{p} dt = r_i^f dt - E_i\left[\frac{d\Lambda}{\Lambda} \frac{dp}{p}\right]
\]

**Chapter 2 Applying the basic model**

1. The consumption-based model is a complete answer to all asset pricing questions, but works poorly in practice. This motivates other asset pricing models.

2. All asset pricing models amount to different functions for \(m\).
   - Different utility functions.
   - General equilibrium models.
   - Factor pricing models: model marginal utility in terms of other variables directly.
   \(M = a + b_1 F_1 + b_2 F_2 + \ldots\)

**Chapter 3 contingent claims markets**

1. Contingent claims

1. **m is contingent claims prices divided by probabilities**, and \(p = E(mx)\) is a bundling of contingent claims.

2. A contingent claim is a security that pays one dollar in one state \(s\) only tomorrow (Arrow-Debrew securities).

3. If there are complete contingent claims, a discount factor exists, and it is equal to the contingent claim price divided by probabilities.
\[ P(x) = \sum_s pc(s)x(s) = \sum_s \pi(s)\left(\frac{pc(s)}{\pi(s)}\right)x(s), \] where \( pc(s) \) is the price of A-D security in state \( s \).

\[ \Rightarrow m(s) = \frac{pc(s)}{\pi(s)} \] called state-price density.

4. **Risk-neutral probabilities:**
   
   let \( \pi^*(s) = R^f m(s)\pi(s) = R^f pc(s) \), where \( R^f = 1/E(m) = 1/\sum pc(s) \),

   \[ R^f \] is gross return.

   - We can think of asset pricing as if agents are all risk neutral, but with probabilities \( \pi^* \) in the place of true probabilities \( \pi \). The probabilities \( \pi^* \) give greater weight to states with higher than average marginal utility \( m \).
   - Risk aversion is equivalent to paying more attention to unpleasant states, relative to their actual probabilities of occurrence. People who report high subjective probabilities of unpleasant events may simply be reporting the risk neutral probabilities or the product \( m^* \pi \): pay a lot of attention to contingencies that are either highly probable or that are improbable but have disastrous consequences.
   - In continuous time under risk neutral \( \Rightarrow \) increase the drift of each price process by its covariance with the discount factor:

\[ \frac{dp}{p} = (\mu^p + \sigma^p \sigma^\Lambda)dt + \sigma^p dz = \mu^{\pi^*} dt + \sigma^p dz \]

\[ \frac{d\Lambda}{\Lambda} = \mu^\Lambda dt \]

\[ E_t\left(\frac{dp}{p}\right) = \mu^{\pi^*} dt \]

---

**Chapter 4**  
**The Discount factor**

I. **Introduction:**

1. **There is a** discount factor that prices all the payoffs by \( p = E(mx) \) iff the **law of one price holds**.
2. **There is a positive** discount factor that prices all the payoffs by \( p = E(mx) \) iff **there are no arbitrage opportunities**.
3. The above two show that we can use discount factor without implicitly assuming about utility functions, aggregation, complete markets, and so on.

II. **Law of one price**

1. The law of one price implies that a discount factor exists: There exists a unique \( x^* \) in \( X \) such that \( p = E(x^*x) \) for all \( x \in X \) = space of all available payoffs. Furthermore, **for any valid discount factor** \( m \), \( x^* = \text{proj}(m|X) \).
2. It says that there is a unique \( x^* \) in \( X \). There may be many other discount factors \( m \) not in \( X \). Markets are incomplete in that contingent claims to lots of states of nature are not available. We found that the law of one price implies a linear pricing function, and a linear pricing function implies that there exists at least one and usually many discount factors even in an incomplete market.

III. No Arbitrage and positive discount factors
1. In complete markets, no arbitrage implies that there exists a unique \( m > 0 \) such that \( p = E(mx) \), because no arbitrage implies the law of one price.
2. No arbitrage implies the existence of a strictly positive discount factor, \( m > 0 \) in each state, \( p = E(mx) \), for any \( x \in X \). It doesn’t say that \( m > 0 \) is unique and discount factors may also exist that are not strictly positive.

Chapter 5 Mean-Variance Frontier and Beta Representation

I. Expected return-beta representation
1. Time-series regression definition of betas:
   \[
   R_i^t = a_i + \beta_{i,a} f_i^a + \beta_{i,b} f_i^b + \ldots + \varepsilon_i^a, \quad t = 1, 2, \ldots, T.
   \]
   The fact that the betas are regression coefficients is crucially important. If the betas are also free parameters, then there is no content to the model. The betas cannot be asset-specific or firm-specific characteristics, such as the size of the firm or book to market ratio. The proper betas should drive out any characteristics in cross-sectional regressions. Only if asset returns depend on how you behave, not who you are – on betas rather than characteristics – can a market equilibrium survive such simple repackaging scheme.

2. Beta-pricing model:
   \[
   E(R_i^t) = \gamma + \beta_{i,a} \lambda_a + \beta_{i,b} \lambda_b + \ldots, \quad i = 1, 2, \ldots, N.
   \]
   - The time series regressions will have a different intercept \( a_i \) for each return \( i \), while the intercept \( \gamma \) is the same for all assets in the beta pricing equation. The beta pricing equation is a restriction on expected returns, and thus imposes a restriction on intercepts in the time-series regression.
   - In the special case that the factors are themselves excess returns, the restriction is particularly simple: the time-series regression intercepts should all be zero. In this case, we can avoid the cross-sectional regression entirely, since there are no free parameters left.

II. Mean-variance frontier: intuition and Lagrangian Characterization
1. In Chapter 1, we derived a similar wedge-shaped region as the set of means and variances of all assets that are priced by a given discount factor. This chapter is about incomplete market, so we think of a mean-variance frontier generated by a given set of assets, typically less than complete.
\[ A = E' \sum^{-1} E; B = E' \sum^{-1} 1; C = 1' \sum^{-1} 1 \]

2. \( \text{Var}(R^p) = \frac{C \mu^2 - 2B \mu + A}{AC - B^2}, w = \sum^{-1} \frac{E(C \mu - B) + 1(A - B \mu)}{AC - B^2} \)

where \( E = E(R) \), \( w \) is portfolio weights.

3. minimum-variance portfolio: minimize var over \( \mu \). \( \mu_{\text{minvar}} = B/C \), the weights are

\[ w = \sum^{-1} 1/1' \sum^{-1} 1 \]

- The frontier is spanned by any two frontier returns.

III. An orthogonal characterization of the Mean-variance frontier

1. Let \( R^* \) be the return corresponding to the payoff \( x^* \) that can act as the discount factor. \( R^* = \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})} \)

2. Let \( R^{e\star} = \text{proj}(1| R^x) \). \( E(R^{e\star}) = E(1xR^{e\star}) = E[\text{proj}(1| R^{e\star})xR^{e\star}] = E(R^{e\star} R^{e\star}) \).

3. Theorem: every return \( R_i \) can be expressed as \( R^i = R^* + w^i R^{e\star} + n^i \), where \( w^i \) is a number; and \( n^i \) is an excess return with the property \( E(n^i) = 0 \). The three components are orthogonal, \( E(R^* R^{e\star}) = E(R^* n^i) = E(R^{e\star} n^i) = 0 \)

4. Theorem: \( R_{mv} \) is on the mean-variance frontier iff \( R_{mv} = R^* + wR^{e\star} \)

IV. Properties of \( R^*, R^{e\star} \) and \( x^* \)

1. \( E(R^{e\star 2}) = 1/E(x^{*2}) \), \( R^* = \frac{x^*}{E(x^{*2})} \)

2. \( E(R^{e\star 2}) = E(R^* R) \), use (***)

3. \( E(R^c) = E(R^{e\star} R^c) \), \( R^{e\star} \) represents means on \( R^c \) via an inner product in the same way that \( x^* \) represents prices on \( X \) via an inner product. \( R^{e\star} \) is orthogonal to planes of constant mean in \( R^c \) as \( x^* \) is orthogonal to planes of constant prices.

4. if a risk-free rate is traded. \( R_f = \frac{1}{E(x^*)} = \frac{E(R^{e\star 2})}{E(R^*)} \)

5. \( R^* \) and \( R^{e\star} \) are orthogonal, \( R^* \) is orthogonal to any excess return.

6. \( R^{e\star} = 1 - \text{proj}(1| R^*) = 1 - R^*/R^f \rightarrow R_f = R^* + R_f R^{e\star} \), \( R^* \) and \( R^{e\star} \) are orthogonal and together span \( X \), because any return can be written like in (***)

V. Mean-variance frontiers for discount factors: The Hansen-Jagannathan Bounds

1. The mean-variance frontier of all discount factors that price a given set of assets is related to the mean-variance frontier of asset returns by
\[
\frac{\sigma_m}{E(m)} \geq \left| \frac{E(R^e)}{\sigma(R^e)} \right|
\]

2. and hence

\[
\min \left\{ \frac{\sigma_m}{E(m)}, \text{all } m \text{ that } x \in X \right\} = \max \left\{ \frac{E(R^e)}{\sigma(R^e)}, \text{all excess returns } R^e \in X \right\}
\]

Chapter 6  Relation between discount factors, betas and mean-variance frontiers

I. From Discount factors to beta representation.
1. In chapter 1, I showed how mean-variance and a beta representation follow from
   \( p = E(mx) \) in a complete markets setting. Here, I discuss the connections in both
   directions and in incomplete markets.
2. All three representations are equivalent.
3. theorem: 1=E(mRi) implies an expected return-beta model with \( x^*=\text{proj}(m\mid X) \) or
   \( R^* = x^* \) as factors, e.g.
   \[
   E(R^i) = \gamma + \beta_{i,x} \cdot \lambda_x \quad \text{and}
   \]
   \[
   E(R^i) = \gamma + \beta_{i,R^*} [E(R^*) - \gamma]
   \]

II. From Mean-variance frontier to a discount factor and beta representations.
1. There is a discount factor of the form \( m = a + bR^{mv} \) iff \( R^{mv} \) is on the mean-
   variance frontier, and \( R^{mv} \) is not the risk-free rate.

III. Factor models and discount factors.
Beta pricing models are equivalent to linear models for the discount factor \( m \),
   \[
   E(R^i) = \alpha + \lambda_i \beta_i \Leftrightarrow m = a + b' f
   \]

IV. Three risk-free rate analogues:
1. **Zero-beta return**: a mean variance efficient return that is uncorrelated with
   another given mean variance efficient return.
   - zero beta return for \( R^* \) denote \( R^\gamma \) (since one can calculate zero-beta rates for
     returns other than \( R^* \) and they are no the same as the zero-beta rate for \( R^* \). the
     zero-beta rate on the “market portfolio” will generally be different from the
     zero-beta rate on \( R^* \). The zero beta rate \( \gamma = E(R^\gamma) \).
   - \( R^\gamma = R^* + \frac{\text{var}(R^*)}{E(R^*)E(R^{e*})} R^{e*}, \gamma = E(R^\gamma) = \frac{E(R^{e*})}{E(R^*)} = \frac{1}{E(x^*)}, \) the
     zero-beta rate is found graphically in mean-standard deviation space by
     extending the tangency at \( R^* \) to the vertical axis. It is also the inverse of the
     price that \( X^* \) and \( R^* \) assign to the unit payoff. **This is property of any return on
     the mean variance frontier**: the expected return on an asset uncorrelated with the
     mean variance efficient asset (a zero-beta asst) lies at the point so constructed.

\[
R_{\text{min.var.}}^{\text{min.var.}} = R^* + \frac{E(R^*)}{1 - E(R^e*)} R^e*, \quad E(R_{\text{min.var.}}^{\text{min.var.}}) = \frac{E(R^*)}{1 - E(R^e*)}
\]

- the minimum variance return retains the nice property of the risk-free, that its weight on \(R^e*\) is the same as its mean.

3. **constant-mimicking portfolio return** is the return on the payoff “closes” to the unit payoff.

- when there is a risk-free rate, it is the rate of return on a unit payoff. \(R^f = 1/p(1)\). When there is no risk-free rate, we can define constant-mimicking portfolio return as:

\[
\overline{R} = \frac{\text{proj}(1 \mid X)}{p[\text{proj}(1 \mid X)]} = R^* + \frac{E(R^*)}{E(R^e*)}
\]

- We can find a discount factor from any mean variance efficient return except the constant-mimicking return.
- We can find a beta representation from any mean variance efficient return except the minimum-variance return.

### Chapter 8 Conditioning information

I. **Scaled payoffs**

\[
p_t = E_t[m_{t+1} x_{t+1}, z_t], \quad p_t = E_t[m_{t+1} x_{t+1} z_t]
\]

\[
p_t = E_t[m_{t+1} x_{t+1}] \Rightarrow E(z_t, p_t) = E[m_{t+1} x_{t+1} z_t]
\]

\(\Rightarrow\) unconditional expectation: \(E(z_t, p_t) = E[m_{t+1} x_{t+1} z_t]\) this offers a simple view of how to incorporate the extra info in conditioning info: add managed portfolio payoffs, and proceed with unconditional moments as if conditioning info did not exist.

II. **Sufficiency of adding scaled returns.**

1. \(E(y_{t+1} \mid I_t)\) is equal to a regression forecast of \(y_{t+1}\) using every variable \(z_t\) in \(I_t\).

   “every random variable” means every variable and every nonlinear transformation of every variable.

2. The word projection and \(\text{proj}(y_{t+1} \mid z_t)\) are used to distinguish the best forecast of \(y_{t+1}\) using only linear combinations of \(z_t\).

3. Checking the expected price of all managed portfolios if, in principle, sufficient to check all the implications of conditioning info.

\[
E(z_t, p_t) = E[m_{t+1} x_{t+1} z_t], \forall z_t \in I_t \Rightarrow p_t = E_t[m_{t+1} x_{t+1} \mid I_t]
\]

III. **Conditional and unconditional models**
1. A conditional factor model doesn’t imply a fixed-weighted or unconditional factor model because the weights may vary:

- $m_{t+1} = b_{f_{t+1}}$, $p_t = E_t[m_{t+1}x_{t+1}]$ does not imply that $3b$ s.t.

$$m_{t+1} = b_{f_{t+1}}, E(p_t) = E[m_{t+1}x_{t+1}]$$

- $E_t(R_{t+1}) = \beta_t \lambda_t$, does not imply $E(R_{t+1}) = \beta \lambda$

- Conditional mean variance efficiency does not imply unconditional mean variance efficiency.

The converse statements are true, if managed portfolios are included.

2. Define the conditional mean-variance frontier as the set of returns that minimize $\text{var}(R_{t+1})$ given $E(R_{t+1})$. Define the unconditional mean-variance frontier as the set of returns including managed portfolio returns that minimize $\text{var}(R_{t+1})$ given $E(R_{t+1})$.

- If a return is on the unconditional mean-variance frontier; it is on the conditional mean-variance frontier.

IV. Scaled factors: a partial solution

1. To express the conditional implications of a given portfolio returns, all you have to do is include some well-chosen scaled or managed portfolio returns, and then pretend you never heard about conditioning info.

2. Some factor models are conditional models, and have coefficients that are functions of investors’ info sets. In general, there is no way to test such models, but if you are willing to assume that the relevant conditioning info is well summarized by a few variables, then you can just add new factors, equal to the old factors scaled by the conditioning variables, and again forget that you ever heard about conditioning info.

Chapter 9 Factor pricing models

1. CAPM

1. The CAPM model: $m = a + bR^w$, $R^w$ = wealth portfolio return. We need to find assumptions that defend which factors proxy for marginal utility, and assumptions to defend the linearity between $m$ and the factor.

- CAPM does not require normal distributions, if one is willing to swallow quadratic utility instead.

2. Two period quadratic utility, investors with no labor income.

3. Exponential utility, normal distribution;

$$u(c) = -e^{-ac}$$ and a normally distributed set of returns also produces the CAPM.

- Assume consumption is normally distributed ($y, yf$: amount of wealth),

- let $c = yfR + yfW = yf + yf1$, max $E(u(c))$ w/ respect to $y, yf$. 

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4. infinite horizon, quadratic utility, and iid returns;
5. log utility.
6. **Stein’s Lemma**, if f, R are bivariate normal, g(f) is differentiable and \( E|g'(f)| < \infty \), then \( \text{Cov}[g(f), R] = E[g'(f)]\text{cov}(f,R) \).
7. **The quadratic utility CAPM and the log utility CAPM should apply to all payoffs**: stocks, bonds, options, contingent claims, etc. Rubinstein (1976) **shows that the log utility CAPM delivers the B-S option**. However, if we assume normal return distributions to obtain a linear CAPM in discrete time, we can no longer hope to price options, since option returns are nonnormally distributed.

II. **ICAPM: any ‘state variable’ \( z_t \) can be a factor**. The ICAPM is a linear factor model with wealth and state variables that forecast changes in the distribution of future returns of income.
1. ICAPM generates linear discount factor models: \( m_{t+1} = a + b' f_{t+1} \), in which the factors are “state variables” for the investor’s consumption-portfolio decision.
2. Let \( c_t = g(z_t) \), we can write: \( m_{t+1} = \beta \frac{V_w(W_{t+1}, Z_{t+1})}{V_w(W_t, Z_t)} \), or in continuous time:
\[
 m_t = \Lambda_t = e^{-\delta t} V_w(W_t, Z_t) \Rightarrow \text{get } \frac{d\Lambda_t}{\Lambda_t} \text{ and plug into basic pricing equation.} \Rightarrow \text{ICAPM.} 
\]
Chapter 19  Term structure of interest rates

I. Definitions and notation

\( P_t^{(N)} \): price of \( N \) period zero-coupon bond at \( t \)

\( p_t^{(N)} \): log price of \( N \) period zero-coupon bond at \( t \)

\[ y^{(N)} = -\frac{1}{N} \ln P^{(N)} = \text{log yield} \]

\[ hpr_{t+1}^{(N)} = p_t^{(N+1)} - p_t^{(N)} = \text{log holding period return.} \]

\[ hpr = \frac{dP(N, t)}{P} - \frac{1}{N} \frac{\partial P(N, t)}{\partial N} dt = \text{instantaneous return} \]

\[ f_t^{(N \rightarrow N+1)} = p_t^{(N)} - p_t^{(N+1)} = \text{log forward rate} \]

\[ f(N, t) = -\frac{1}{P} \frac{\partial P(N, t)}{\partial N} = \text{instantaneous forward rate} \]

II. Continuous-time term structure models.

1. A time-series model for the discount factor, typically in the form:

\[ \frac{d\Lambda}{\Lambda} = -rdt - \sigma_{\Lambda}(.)dz \]

\[ dr = \mu_r(.)dt + \sigma_r(.)dz \]

2. Vasicek: \( \frac{d\Lambda}{\Lambda} = -rdt - \sigma_{\Lambda}dz \)

\[ dr = \phi(\bar{r} - r)dt + \sigma_r dz \]

3. Cox-Ingersoll-Ross: \( \frac{d\Lambda}{\Lambda} = -rdt - \sigma_{\Lambda}\sqrt{\Lambda}dz \)

\[ dr = \phi(\bar{r} - r)dt + \sigma_r \sqrt{r} dz \]

III. Solve the discount factor model forward and take expectations, to find bond prices

\[ P_t^{(N)} = E_t(\frac{\Lambda_{t+N}}{\Lambda_t}) \]

1. Expectation approach

2. Differential equation approach