Expected Shortfall (ES)
Reminders on conditional expectations
Expected shortfall
Expected shortfall with bonds
Advantages and disadvantages
Reminders on conditional expectations

- Expected shortfall
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Reminders on conditional expectations
\[ E(X) = \sum_{i=1}^{T} p_i x_i \]  

(4.3.1)

\[ A = \{ i \mid x_i < z \} \]  

(4.3.2)

\[ E(X \mid X < z) = \frac{\sum_{i \in A} p_i x_i}{\sum_{i \in A} p_i} \]  

(4.3.3)
Continuous random variables

\[ E(X) = \int_{-\infty}^{\infty} x f_x(x) \, dx \quad (4.3.4) \]

\[ p = \Pr(X \leq z) = \int_{-\infty}^{z} f_x(x) \, dx \quad (4.3.5) \]

\[ E(X|X \leq z) = \frac{1}{p} \int_{-\infty}^{z} x f_x(x) \, dx \quad (4.3.6) \]
Expected shortfall

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Advantages and disadvantages
**Expected shortfall:** Expected loss given that VaR loss is exceeded.

- More information on left tail
- Also, known as,
  - Expected tail loss (ETL)
  - Conditional Value-at-Risk (CVaR)
For a continuously distributed loss

\[ ES(p) = -E(Q | Q \leq -\text{VaR}(p)) \]  \hspace{1cm} (4.3.7)

\[ p = \int_{-\infty}^{-\text{VaR}(p)} f_q(x) \, dx \]  \hspace{1cm} (4.3.8)

\[ ES = -E(Q | Q \leq -\text{VaR}(p)) = -\frac{1}{p} \int_{-\infty}^{-\text{VaR}(p)} x f_q(x) \, dx \]  \hspace{1cm} (4.3.9)
A quick formula

• For certain restricted portfolios expected shortfall can be calculated with tables.
• Assume P/L is normal, mean 0, std 1
• \( \phi(x) \) is normal density, \( \Phi(x) \) is normal CDF

\[
ES = \frac{\phi(\Phi^{-1}(p))}{p}
\]

(4.3.10)

See Daníelson, 5.3.4 for a derivation.
# set tail probs
p = np.array([0.5, 0.1, 0.05, 0.025, 0.01])
# find VaR with inverse CDF
VaR = -stats.norm.ppf(p)
# Now ES with fancy formula
ES = stats.norm.pdf(stats.norm.ppf(p))/p
These are for the example above with a normal distribution, mean zero, and std. of 1.

<table>
<thead>
<tr>
<th>p</th>
<th>VaR</th>
<th>ES</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.282</td>
<td>1.755</td>
</tr>
<tr>
<td>0.05</td>
<td>1.645</td>
<td>2.063</td>
</tr>
<tr>
<td>0.01</td>
<td>2.326</td>
<td>2.665</td>
</tr>
<tr>
<td>0.001</td>
<td>3.090</td>
<td>3.367</td>
</tr>
</tbody>
</table>

This seems similar, but what about other distributions? *We will get to this soon.*
Expected shortfall with bonds
Return to bond example

- **Bond**
  - 1 year maturity
  - Principal = 100 = purchase price
  - Interest = 5%
  - No default: pays 105

- **Default probability** = 0.0125

- If there is a default, bond returns \([0, 100]\) uniform (recovery)
Default density and $\text{VaR}(0.01)$

\[ 100 \times 1.25 \times 10^{-4} = 0.0125 \quad 80 \times 1.25 \times 10^{-4} = 0.01 \]
What is the expected shortfall?

- $ES(0.01)$ is the expected loss conditional on going past the 0.01 VaR level.
- Range of bond values: Uniform $[0, 80]$
- Expected value $= 40$
- Expected loss $= 40 - \text{Principal} = 40 - 100 = -60$
- Expected shortfall $= -(\text{Expected loss}) = 60$

\[
ES = -\left[\frac{1}{p} \int_{0}^{80} xf(x)dx - 100\right], \quad f(x) = 1.25 \times 10^{-4}, \quad p = 0.01
\] (4.3.11)

or more formally,

\[
ES = \frac{-1}{p} \int_{-100}^{-20} xf(x)dx, \quad f(x) = 1.25 \times 10^{-4}, \quad p = 0.01
\] (4.3.12)
Can our sneaky trader manipulate ES?

Return to single bond example with insurance

- Investor holds single bond as defined before
- Investor sells insurance on bond defaults to others
- Only insure in states where recovery is less than 80 (of 100)
- Pay out to take other investor back to 80 in all these states
**Implementing the strategy**

- **Sell** insurance on all losses beyond 20
- **What happens**
  
  1. No default: earn insurance premium with no default, $105 + x$
  
  2. Default ($> 80$): Loss less than 20, no change
  
  3. Default ($< 80$): All defaults recovering less than 80 trigger insurance payments

  $\Rightarrow$ Investor receives: $X - (80 - X) = 2X - 80$

  Portfolio values change: $80 \rightarrow 80$, $0 \rightarrow -80$, **range** = $[80, -80]$

  $\Rightarrow$ What is the loss in this range? $-[80 - 100, -80 - 100] = [20, 180]$

  $\Rightarrow$ Previously it was: $-[80 - 100, 0 - 100] = [20, 100]$
The strategy in a picture

Before insurance

After insurance
What is the expected shortfall now?

- \( \text{ES}(0.01) \) is the expected loss conditional on going past the 0.01 VaR level.
- Range of portfolio values: Uniform \([-80, 80]\)
- Expected value = 0
- Expected loss = \(0 - \text{Principal} = 0 - 100 = -100\)
- Expected shortfall = -(Expected loss) = 100

\[
\text{ES} = -\left[ \frac{1}{p} \int_{-80}^{80} x f(x) dx - 100 \right], \quad f(x) = (1.25/2) \times 10^{-4}, \quad p = 0.01
\]

or more formally,

\[
\text{ES} = \frac{-1}{p} \int_{-180}^{-20} x f(x) dx, \quad f(x) = (1.25/2) \times 10^{-4}, \quad p = 0.01
\]
Comparisons between ES and VaR

- VaR didn’t change: 20 in both cases
- ES increased from 60 to 100
- It is much more difficult to hide changes to the left tail with ES
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ES advantages and disadvantages

- Advantages
  - More info on extreme losses
  - Less easily manipulated
  - ES is subadditive
ES advantages and disadvantages

- **Advantages**
  - More info on extreme losses
  - Less easily manipulated
  - ES is **subadditive**

- **Disadvantages**
  - More difficult to calculate (depends on tail data)
  - More difficult to explain
  - Institutions all use VaR
  - Backtesting difficult
Backtesting

- Tests to see if risk measures were working in past
- VaR
  - Observe VaR exceedances
    \[ \hat{p} = \Pr(Q \leq \text{VaR}(p)) \] (Count)
    - Fraction of losses greater than VaR
    - Should be \( p \)
- ES
  - This is much trickier
  - Were the conditional expectations correct in the past?
    \[ ES = -E(Q|Q \leq -\text{VaR}(p)) \]
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