A Harder-Narasimhan theory for Kisin modules

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Theorem (Wiles, Taylor-Wiles, BCDT)

*Any elliptic curve* $E/\mathbb{Q}$ *is modular.*

A key input into Wiles’ method and subsequent improvements is a good understanding of Galois deformation spaces at $\ell = p$. 
Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $\Gamma_K$ be the absolute Galois group of $K$. Fix
\[
\bar{\rho} : \Gamma_K \rightarrow \text{GL}_n(F_p).\]
There is a universal deformation space $D_{\bar{\rho}}$ represented by a quotient of a power-series ring $R_{\bar{\rho}}$ over $\mathbb{Z}_p$ (when $\bar{\rho}$ is absolutely irreducible).
If $E$ is an elliptic curve over $K$ with good reduction, then $E[p^n]$ is a finite flat group scheme over $O_K$ for all $n$. The representation of $\Gamma_K$ on the $p^n$-torsion points is called flat.

The flat deformation space $D_{\rho}^{fl}$ is the subspace of $D_{\rho}$ of representations that come from finite flat group schemes over $O_K$. 
What are the connected components of $D_{\rho}^{\text{fl}}[1/p]$?

- (Ramakrishna) When $n = 2$ and $K = \mathbb{Q}_p$, then $D_{\rho}^{\text{fl}}[1/p]$ is connected.
- When $n = 2$, we have full description of connected components for any $K$ by work of Kisin, Imai, Gee, and Hellmann.
- When $n > 2$, the question is open in general (unless $K$ is mildly ramified).
Theorem (Kisin)

There is a projective variety $X_{\overline{\rho}}$ over $\mathbb{F}_p$ such that $X_{\overline{\rho}}(\mathbb{F})$ is the set of finite flat group schemes $\mathcal{G}$ over $\mathcal{O}_K$ such that $\mathcal{G}(\overline{K}) \cong \overline{\rho} \otimes_{\mathbb{F}_p} \mathbb{F}$.

Application

Connected components of $D^\text{fl}_{\overline{\rho}}[1/p]$ are related to the connected components of $X_{\overline{\rho}}$. 
Kisin modules

Definition

Assume $K / \mathbb{Q}_p$ is totally ramified of degree $e$ and $\mathbb{F}$ is a finite field. Let $\varphi : \mathbb{F}[[u]] \to \mathbb{F}[[u]]$ be the homomorphism sending $u \mapsto u^p$. A Kisin module of rank $n$ over $\mathbb{F}$ is a finite free $\mathbb{F}[[u]]$-module $M_\mathbb{F}$ with a semilinear map

$$\phi : M_\mathbb{F} \to M_\mathbb{F}$$

such that the cokernel (of the linearization) is killed by $u^e$.

Theorem (Kisin)

The category of Kisin modules over $\mathbb{F}$ is anti-equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ with an $\mathbb{F}$-action.
The generic fiber of \((\mathcal{M}, \phi)\) is \((\mathcal{M}[1/u], \phi[1/u])\). (This is an étale \(\mathbb{F}((u))\)-module).

The degree of \((\mathcal{M}, \phi)\) is \(\frac{1}{e} \dim_{\mathbb{F}} \text{coker}(\phi)\).

The slope is \(\mu(\mathcal{M}) := \deg(\mathcal{M}) / \text{rank}(\mathcal{M})\).

This was inspired by Fargues’ (2010) theory of Harder-Narasimhan filtrations for finite flat group schemes.
Theorem (L.-W. E.)

The function $\mu$ defines a HN-theory on the category of Kisin modules. In particular, any $\mathcal{M}$ has a canonical HN-filtration

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_k = \mathcal{M}$$

by strict subobjects such that $\mathcal{M}_{i+1}/\mathcal{M}_i$ is semi-stable and $\mu(\mathcal{M}_i/\mathcal{M}_{i-1}) < \mu(\mathcal{M}_{i+1}/\mathcal{M}_i)$.

- The HN-filtration generalizes the connected-etale sequence for finite flat group schemes.
- The HN-polygon is the concave polygon with breakpoints $(\text{rank}(\mathcal{M}_i), \text{deg}(\mathcal{M}_i))$. 

Kisin varieties

Definition

For $\nu = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{Z}$ and $a_{i+1} \geq a_i$, a Kisin module $(M, \phi_M)$ over $\mathbb{F}$ of rank $n$ has **Hodge type** $\nu$ if there exists a basis $\{e_i\}$ of $M$ such that $u^{a_i} e_i$ generates the image of $\phi_M$.

Definition

Let $(M_{\bar{\rho}}, \phi)$ be the étale $\mathbb{F}_p((u))$-module of rank $n$ attached to $\bar{\rho}$. The closed Kisin variety has points given by

$$X^\nu_{\bar{\rho}} = \{ M[1/u] \cong M_{\bar{\rho}} \mid M \text{ has Hodge type } \leq \nu \}.$$ 

It is a projective scheme over $\mathbb{F}_p$.

Remark

*This is subspace of $X_{\bar{\rho}}$ if $0 \leq a_i \leq e$ for all $i$.***
Stratification

**Theorem (L.-W. E.)**

There is a stratification

$$\bigcup_{P} X_{\rho, P}^{\nu} = X_{\rho}^{\nu}$$

by locally closed subschemes indexed by concave polygons $P$ such that the points of $X_{\rho, P}^{\nu}$ are the Kisin modules with HN-polygon $P$.

**Remark**

For any point in $X_{\rho}^{\nu}$, the HN-polygon lies above the Hodge polygon $\nu$. Hence, there are a finite number of such strata.
Explanation

In the following slides, for different Hodge polygons $\nu$, we draw the set of possible HN-polygons.

- For any $\bar{\rho}$ of the appropriate dimension, the strata of $X^\nu_{\bar{\rho}}$ will be indexed by this finite set of polygons.
- The Hodge polygon $\nu$ appears in black.
- We color the polygons the same if they share the same segments in common with the Hodge polygon.
- Only strata with the same color can occur on the same connected component (i.e., the union of the strata with same color is open and closed in $X^\nu_{\bar{\rho}}$).

Remark

For any particular $\bar{\rho}$, many of the strata could be empty. For example, if $\bar{\rho}$ is irreducible, then only the constant slope stratum will appear.
Components for $GL_2, \nu = (0, 3)$
Components for $GL_3$, $\nu = (0, 0, 1)$
Components for $\text{GL}_3$, $\nu = (-1, 0, 1)$
Components for $\text{GSp}_4$, $\nu = (-2, -1, 1, 2)$

Non-Ordinary

Partially Ordinary

Ordinary
Conjecture

Let $M$ and $N$ be Kisin modules over $F$. If $M$ and $N$ are semistable, then

$$M \otimes_F N$$

is semistable of slope $\mu(M) + \mu(N)$.

Theorem (L.-W. E.)

The tensor product theorem holds when $M[1/u]$ and $N[1/u]$ are irreducible (with a few technical assumptions).

Application

Study Kisin varieties for reductive groups $G$ and $G$-valued flat deformation rings.