Topics in Consumption Theory

1. Graphs and Motivation
2. Intertemporal Elasticity of Substitution
3. The General Model
4. Consumption as a Random Walk
5. Certainty Equivalence
6. Precautionary Saving
Intertemporal Elasticity of Substitution

• Let

\[
\sigma(c(t), c(s)) \equiv -\frac{\Delta\% \left[ \frac{c(t)}{c(s)} \right]}{\Delta\% \left[ \frac{u'(c(t))}{u'(c(s))} \right]}
\]

• Letting \( s \to t \), we obtain

\[
\sigma(c(t)) = \frac{u'(c)}{cu''(c)},
\]

the instantaneous elasticity of substitution

• High elasticity: consumption responds a lot to \( \Delta r \)

• Inverse of \( \sigma(c) \) is the coefficient of relative risk aversion (CRRA)
An Useful Family of Utility Functions

- Often use constant intertemporal elasticity of substitution (CES) utility function.

\[ u(c) = \begin{cases} 
\frac{c^{1-\theta} - 1}{1-\theta} & \theta > 0, \theta \neq 1 \\
\ln(c) & \theta = 1 
\end{cases} \]

- \( \sigma(c) = \frac{1}{\theta} \), independent of \( c \)
- CRRA = \( \theta \).
The General Model

Consider the following model:

$$\max_{c_t} E_0 \sum_{t=0}^\infty \beta^t u(c_t)$$

subject to:

$$a_{t+1} = (1 + r_t)(a_t + y_t - c_t)$$

where

- $a_t =$ beginning of period $t$ assets
- $y_t =$ period $t$ income
- $c_t =$ period $t$ consumption
- $r_t =$ period $t$ return of assets
- $\beta \equiv \frac{1}{1 + \delta}$ subjective discount factor
- $\delta =$ subjective discount rate (or rate of pure time preference)

with $u(c_t)$ increasing and concave. $y_t$ and $r_t$ may be random variables.
Bellman Formulation

\[ V(a, y, r) = \max_c \{ u(c) + \beta EV(a', y', r') \} \]

subject to

\[ a' = (1 + r)(a + y - c) \]

- Let’s make our life easier
  - Set \( y_t = \bar{y} = 0 \) (diversifiable labor income case)
  - Set \( r_t = r \) (constant interest rate)

- Now there is no uncertainty
Bellman equation can be written as

\[ V(a) = \max_c \left\{ u(a - \frac{a'}{1 + r}) + \beta V(a') \right\} \]

first-order condition is

\[ u' \left( a - \frac{a'}{1 + r} \right) \frac{-1}{1 + r} + \beta V'(a') = 0 \]

The envelop condition is

\[ V'(a) = u' \left( a - \frac{a'}{1 + r} \right) \]

These two conditions imply the following Euler equation

\[ u'(c_t) = (1 + r) \beta u'(c_{t+1}) \]

or

\[ u'(c_t) = \frac{1 + r}{1 + \delta} u'(c_{t+1}) \]
• Hence

\[ \delta = r \Rightarrow c_t \text{ constant} \]

\[ \delta > r \text{ (impatient)} \Rightarrow c_t \text{ downward sloping} \]

\[ \delta < r \text{ (patient)} \Rightarrow c_t \text{ upward sloping} \]

• If we assume that \( u \) is CES:

\[ \frac{c_{t+1}}{c_t} = (\beta(1 + r))^{1/\theta} \]

• Hence

\[ \Delta \ln(c_{t+1}) = \frac{\ln(\beta) + \ln(1 + r)}{\theta} \approx \frac{r - \delta}{\theta} \]
Guess and Verify a Solution

- Guess solution of the form
  \[ V(a) = G + H \ln(a) \]

- Can show \( H = \frac{1}{1-\beta} \) and \( c = \frac{\delta}{1+\delta} A \)

- Consumption is linear in wealth.

- Consumption/saving decision depends only on subjective discount rate \( \delta \), not on \( r \).
Certainty Equivalence

Four Assumptions

1. Quadratic utility
\[ u(c) = c - \frac{b}{2}c^2 \]
   • Thus marginal utility is linear in consumption.
   • Simplifies the math considerably
   • Unfortunate consequences: 1) \( u'(0) < \infty \) so negative consumption may be optimal; 2) may hit bliss point after which \( u'(c) < 0 \).

2. \( r_t = r = \delta \) thus \( \beta = \frac{1}{1+r} \).

3. \( y_t \) an i.i.d. random variable, observe before \( c_t \) decision

4. Infinite horizon
• Define $x_t \equiv a_t + y_t$ to be “cash on hand”

• So Bellman equation is

$$V(x) = \max_c \{u(c) + \beta EV(x')\}$$

subject to:

$$x' = (1 + r)(x - c) + y'$$

• Use the first-order condition and the envelop condition to derive the Euler equation:

$$u'(c_t) = \beta(1 + r)E_t u'(c_{t+1})$$

or

$$E_t u'(c_{t+1}) = \frac{1}{\beta(1 + r)}u'(c_t)$$

• Marginal utility of consumption follows a Markov process.
Since utility is quadratic, and $\beta(1+r) = 1$, this Euler equation becomes:

$$c_t = E_t c_{t+1}$$

Let

$$\epsilon_t \equiv C_t - E_{t-1}C_t.$$ 

Then

$$C_t = C_{t-1} + \epsilon_t$$

with

$$E_{t-1}\epsilon_t = 0$$

Hence $c_t$ follows a random walk.

Explicit Expressions for $c_t$ and $\Delta c_t$

- Use inter-temporal budget constraint:

$$a_{t+1} = (1 + r)(a_t + y_t - c_t)$$

recursively to derive

$$\sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t c_t = a_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t y_t.$$ 

- Take $E_0$ on both sides to obtain expected budget constraint.

$$\sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t E_0 c_t = a_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t E_0 y_t.$$ 

- Since $E_0 c_t = c_0$, this expression becomes

$$c_0 = \frac{r}{1 + r} \left( a_0 + E_0 \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t y_t \right)$$
• In general

\[ c_t = \frac{r}{1 + r} \left( a_t + \sum_{s=0}^{\infty} \left( \frac{1}{1 + r} \right)^s E_t y_{t+s} \right) \]

• Interpretation: Certainty Equivalence
  – Replace stochastic variables by their expected values.
  – Consumption does not change if the variance of \( y \) changes.

• Consumption linear in total wealth
  – constant of proportionality: \( \frac{r}{1+r} \).
  – first term \((a_t)\): financial wealth
  – second term (EPDV of \( y \)): human wealth
• With a bit of patience can show that:

\[
\Delta c_t = \frac{r}{1 + r} \sum_{s=0}^{\infty} \left( \frac{1}{1 + r} \right)^s [E_t y_{t+s} - E_{t-1} y_{t+s}]
\]

(1)

• Hence \( \Delta c \) equals the change in expected annuity value from wealth, i.e. change in permanent income.

• This result can be interpreted as a modern version of the Friedman’s Permanent Income Hypothesis (PIH).

• If we work with finite horizons, it can be interpreted as a modern version of Modigliani’s Life-cycle Theory (LCT).
Some Particular Income Processes: $y_t$ i.i.d.

- Assume

$$y_t + \mu + e_t$$

with $e_t$ i.i.d. with zero mean.

- Then from equation (1) we get

$$\Delta c_t = \frac{r}{1 + r} e_t$$

- Hence

$$\frac{\partial \Delta c_t}{\partial e_t} = \frac{r}{1 + r} << 1.$$ 

- Transitory shocks: consumption adjusts by just the annuity value of the shock.
**Some Particular Income Processes: \( y_t \) follows AR(1)\)**

- Assume \( y_t - \mu = \rho(y_{t-1} - \mu) + e_t \)
- Assume \( 0 \leq \rho < 1 \) denotes the first-order autocorrelation.
- \( \rho = 0 \) corresponds to i.i.d. case (all shocks transitory).
- \( \rho = 1 \) corresponds to random walk (all shocks permanent).
- Then for \( s \geq 0 \)

\[
E_t[y_{t+s} - \mu] = \rho^s(y_t - \mu) \Rightarrow E_t[y_{t+s}] = \mu + \rho^s(y_t - \mu)
\]
• Hence from equation (1)

\[
\Delta c_t = \frac{r}{1 + r} \sum_{s \geq 0} \left( \frac{1}{1 + r} \right)^s (\rho^s(y_t - \mu) - \rho^{s+1}(y_{t-1} - \mu))
\]

\[
= \frac{r}{1 + r} \sum_{s \geq 0} \left( \frac{1}{1 + r} \right)^s \rho^s e_t
\]

\[
= \frac{r}{1 + r - \rho} e_t
\]

• Hence

\[
\frac{\partial \Delta c_t}{\partial e_t} = \frac{r}{1 + r - \rho}.
\]

• Varies from \( \frac{r}{1+r} \) (for \( \rho = 0 \)) to 1 (for \( \rho = 1 \)).

• Consumption responds more to more persistence shocks.

• Adjusts fully to shocks for \( \rho = 1, \Delta c_t = e_t \).
Intuition

- Expected contribution of $e_t$ to $y_{t+s}$ is $\rho^s e_t$.
- Individuals discount future income, hence value $t+s$ income by $\beta^s \rho^s e_t$.
- Hence by certainty equivalence

$$\Delta c_t = \frac{r}{1 + r} \sum_{s \geq 0} \beta^s \rho^s e_t = \frac{r}{1 + r - \rho} e_t$$

- This extend consumption smoothing intuition to stochastic case: increase consumption by annuity value of present discounted value of expected additional income associated with current shock $e_t$. 
Precautionary Saving

• Under certainty-equivalence, more income uncertainty (e.g. a mean preserving spread) does not affect saving ...

• Hmm ....

• It turns out that $u''' > 0$ is necessary and sufficient to obtain precautionary motive for saving
An Informal Derivation

Use two results from decision making under uncertainty

A. Jensen’s Inequality

If

\[ f : \mathcal{R} \rightarrow \mathcal{R} \]

\( f \) is strictly convex

\( x \) is a random variable with \( \text{var}(x) > 0 \).

then \( E[f(x)] > f[E(x)] \)

B. Mean-preserving spread and \( E[f(x)] \)

If

\[ f : \mathcal{R} \rightarrow \mathcal{R} \]

\( f \) is strictly convex

\( y \) is a random variable symmetric w.r.t. 0.

\( y \) is independent of \( x \)

\( z = x + y \) (mean preserving spread)

then \( E[f(z)] > E[f(x)] \)
• Assume $r = \delta = 0$

• Assume mean and variance of $c_{t+1}$ (conditional on $t$) is given exoge-
  nously (this is an informal derivation).

• Consider mean preserving spread on $c_{t+1}$ conditional on $t$.

• Then

\[ E_t[C_{t+1}^{\text{post}}] = E_t[C_{t+1}^{\text{pre}}] \]
\[ Var_t[C_{t+1}^{\text{post}}] > Var_t[C_{t+1}^{\text{pre}}] \]

• Then from Euler equation

\[ u'(c_t) = E_t u'(c_{t+1}). \]
Hence if \( u' \) is convex (\( u''' > 0 \)):

\[
u'(c^p_t) = E_t[u'(c^p_{t+1}) \text{ from Euler equation} \\
> E_t[u'(c^{pre}_{t+1}) \text{ from Property B} \\
> u'(E_t[c^{pre}_{t+1}]) \text{ from Property A}
\]

- Hence

\[c^p_t < E_t[c^{pre}_{t+1}]\]

- By contrast, easy to show that with certainty equivalence:

\[c^{C.E.,post}_t = E_t[c^{C.E.,pre}_{t+1}]\]

- Since we assumed the right hand side of both expressions above are equal, we have shown that precautionary saving, as defined as:

\[s^{precautionary}_t \equiv s^p_t - s^{C.E.,post}_t = c^{C.E.,post}_t - c^p_t\]

is strictly positive when \( u''' > 0 \).

- Similar argument shows, informally, that precautionary saving is negative if \( u''' < 0 \).
What can we say about the sign of $u'''$?

- No theoretical guides for sign of $u'''$.
- Hard to measure $u'''$ directly from micro studies.
- Note: CES utility functions have $u''' > 0$. 