Inter-temporal Choice

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Inter-temporal Choice: Irving Fisher

- Assumptions
  1. no uncertainty
  2. no money, so no inflation
  3. no taxes,

- Consider an environment in which agents live for two periods:
• We assume an individual gets utility from consumption ($C_t$) in each period. He or she is an utility maximizer so he or she wishes to:

$$\max_{C_1, C_2} U(C_1, C_2)$$

• For now we assume he or she can borrow and lend freely across the two periods at an interest rate $r$. The agent faces the following budget constraint:

$$C_1 + \frac{C_2}{1 + r} = Y_1 + \frac{Y_2}{1 + r}$$

• Note everything in this equation is units of “goods in period 1”. The term $\frac{1}{1 + r}$ is relative price of goods between the two periods and converts the units from the second period into the first.

• This budget line equates the present value of lifetime consumption (PVLC) to the present value of lifetime resources (PVLR).
The agent’s constrained maximization problem

• The individual’s problem once again is:

\[
\max_{C_1, C_2} U(C_1, C_2)
\]

subject to:

\[
C_1 + \frac{C_2}{1 + r} = Y_1 + \frac{Y_2}{1 + r}
\]

To solve the problem we substitute the constraint into the objective function so agent’s problem becomes:

\[
\max_{C_2} U \left( Y_1 + \frac{Y_2}{1 + r} - \frac{C_2}{1 + r}, C_2 \right)
\]
• We take the first derivative with respect to $C_2$:

$$U_1(C_1, C_2) \left( \frac{-1}{1 + r} \right) + U_2(C_1, C_2) = 0$$

where $U_1$ is derivative of $U$ with respect to the first argument, and where $U_2$ is derivative of $U$ with respect to the second argument.

• Rearrange terms:

$$\frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = 1 + r$$

• So the agent chooses $C_1$ and $C_2$ such that the marginal rate of substitution between consumption today for consumption tomorrow is equal to the interest rate $1 + r$. 
An example with a specific utility function

- Set $U(C_1, C_2) = \ln(C_1) + \beta \ln(C_2)$ where $\beta < 1$

- Why is $\beta < 1$?
  
  - We assume agents discount future consumption at the rate $\beta$. That is, they prefer current consumption to future consumption all other things held equal.

- We write the agent’s problem as

\[
\max_{C_1, C_2} \ln(C_1) + \beta \ln(C_2)
\]

subject to:

\[
C_1 + \frac{C_2}{1 + r} = Y_1 + \frac{Y_2}{1 + r}
\]
Solving

1. Substitute the constraint into the objective function so the problem becomes:

   \[
   \max_{C_2} \ln \left( Y_1 + \frac{Y_2}{1+r} - \frac{C_2}{1+r} \right) + \beta \ln(C_2)
   \]

2. Take the derivative with respect to \( C_2 \):

   \[
   \frac{1}{Y_1 + \frac{Y_2}{1+r} - \frac{C_2}{1+r}} \left( \frac{-1}{1+r} \right) + \frac{\beta}{C_2} = 0
   \]

3. Do some algebra

   \[
   C_2 = (1 + r) \left( \frac{\beta}{1+\beta} \right) \left( Y_1 + \frac{Y_2}{1+r} \right)
   \]

4. Using the budget constraint to solve for \( C_1 \) yields

   \[
   C_1 = \left( \frac{1}{1+\beta} \right) \left( Y_1 + \frac{Y_2}{1+r} \right)
   \]

5. Saving in this model economy is

   \[
   S = Y_1 - C_1
   = Y_1 - \left( \frac{1}{1+\beta} \right) \left( Y_1 + \frac{Y_2}{1+r} \right)
   \]
So what’s the point of all this math?

- We just derived consumption and saving functions.
- Note consumption in both periods depend on PVLR.
- Really the agent’s problem boils down to how to split total lifetime income between the two periods.
\begin{itemize}
  \item If $S > 0$, this person is a lender, $S < 0$, this person is a borrower.
  \item Note that for ln utility
    \[ \frac{C_2}{C_1} = \frac{(1 + r) \left( \frac{\beta}{1+\beta} \right) \left( Y_1 + \frac{Y_2}{1+r} \right)}{\left( \frac{1}{1+\beta} \right) \left( Y_1 + \frac{Y_2}{1+r} \right)} = \beta(1 + r) \]
  \item So if
    \[ \beta > \frac{1}{1+r} \]
    the agent consumes more in the second period than in the first period.
  \item If
    \[ \beta < \frac{1}{1+r} \]
    the agent consumes more in the first period than in the second period.
  \item If
    \[ \beta = \frac{1}{1+r} \]
    the agent consumes equal amounts in both periods.
\end{itemize}
Irving Fisher’s Two-Period Capital Theory

- Add production to the model.
  Recall our agent lives for two periods: 1 and 2. This agent gets utility from consumption in each period:

  \[
  \max_{C_1, C_2} U(C_1, C_2)
  \]

- But let’s assume for the time being that instead of being handed income each period, this agent has access to a production function:

  \[AF(K)\]

Assume the agent comes into this world with a certain a given capital stock \((K_1)\). The agent produces \(AF(K_1)\) stuff during the initial time period. Of that stuff she can either eat it or save it.
In particular, assume $AF(K) = AK^\alpha$.

There are two ways to interpret this production function.

1. The production function is Cobb-Douglas, and we set $N=1$,

2. We are thinking about a per-capita output $Y/N = AK^\alpha N^{1-\alpha}$

where $k$ is the capital to labor ratio. In this second case we want to think of $C$ as representing per capita consumption and $K$ as representing the capital to labor ratio.
So in period 1, the agent’s budget constraint is:

\[ C_1 + K_2 = AF(K_1) \]

In period 2, there is no saving decision. The agent just eats everything he or she produces:

\[ C_2 = AF(K_2) \]

So the agent’s problem is:

\[
\max_{C_1, C_2} U(C_1, C_2)
\]

subject to:

\[
C_1 + K_2 = AF(K_1)
\]

\[
C_2 = AF(K_2)
\]

Relative price of \( K_2 \) to \( C_1 \) is 1. That is, the price of capital \( p_K \) is 1.
How do we solve this problem?

1. Substitute the constraints into the utility function:

   \[
   \max_{K_2} U(AF(K_1) - K_2, AF(K_2))
   \]

   The choice variable is now \( K_2 \) which is the saving from period 1 to period 2.

2. Get the first-order necessary condition for a maximum by taking a derivative with respect to \( K_2 \):

   \[
   U_1(C_1, C_2)(-1) + U_2(C_1, C_2)AF'(K_2) = 0
   \]

3. Rearrange terms

   \[
   \frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = AF'(K_2)
   \]
This pins down the interest rate

- recall we just derived:

\[ \frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = 1 + r \]

- We can also solve the one period problem of a firm maximizing the value of production (in terms of goods).

\[ \max_K AF(K) - (1 + r)K \]

This implies

\[ AF'(K) = 1 + r \]
The deterministic one-sector growth model

• no labor (marginal utility of leisure set to 0)
• complete depreciation of capital

finite horizon case

\[
\max_{\{c_t, k_{t+1}\}_{t=0}^{T-1}} \sum_{t=0}^{T} \beta^t U(c_t)
\]

subject to

\[
c_t + k_{t+1} = f(k_t)
\]

\[k_0 \text{ given}
\]

\[k_t \geq 0
\]
• substituting the constraint into the utility function yields:

\[
\max_{\{k_{t+1}\}_{0}^{T-1}} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1})
\]

given the initial conditions stated above.

• Taking first-order conditions yields

\[
\beta^t U'(f(k_t) - k_{t+1})(-1) + \beta^{t+1} U'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) = 0
\]

- holds for \( t = 0, 1, 2, \ldots T - 1 \)
- at time \( T, \ k_{t+1} = 0 \)
Pick functional forms

- $U(c_t) = \ln c_t$
- $f(k_t) = A k^\alpha$.

The first order condition then becomes

$$\beta^t \frac{1}{A k_t^\alpha - k_{t+1}}(-1) + \beta^{t+1} \frac{\alpha A k_{t+1}^{\alpha-1}}{A k_{t+1}^\alpha - k_{t+2}} = 0$$
for $t=0,1,2, \ldots T-1$

$k_{T+1} = 0$

- Still have $T$ non-linear equations and $T$ unknowns.
Trick – change the variables

- Define

\[ X_t = \frac{k_{t+1}}{Ak_t^\alpha} \]

so \( X_t \) is the savings rate. Substituting back into the first order condition yields

\[ X_{t+1} = 1 + \alpha \beta - \frac{\alpha \beta}{X_t} \]

- We know \( X_T = 0 \), so we can solve the difference equation by backwards substitution, then use \( k_0 \) to get the capital stock. Working backwards yields:

\[ X_t = \frac{\alpha \beta 1 - (\alpha \beta)^{T-t+1}}{1 - (\alpha \beta)^{T-t+2}} \]
Infinite horizon case

$$\max_{\{k_{t+1}\}_0} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha - k_{t+1})$$

with the transversality condition:

$$\lim_{T \to \infty} \beta^T \left( \frac{\alpha Ak_T^{\alpha-1}}{Ak_T^\alpha - k_{T+1}} \right) k_T = 0$$

- If we take first-order conditions, we get an infinite number of non-linear second-order difference equations and infinite number of unknowns.
- That’s tough to solve.
- But note that in the finite time model that if $T$ is large

$$X_t \approx \alpha \beta$$

or

$$k_{t+1} \approx \alpha \beta Ak_t^\alpha$$
**conjecture:** the solution to this infinite horizon problem has the form

\[ k_{t+1} = h(k_t), \quad t = 0, 1, \ldots \]

In particular

\[ k_{t+1} = \alpha \beta A k_t^\alpha. \]

So this suggests another approach.
• Define a function

\[ V(k_0) = \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha - k_{t+1}). \]

• We can then go forward one period and define:

\[ V(k_1) = \max_{k_{t+1}} \sum_{t=1}^{\infty} \beta^t \ln(Ak_t^\alpha - k_{t+1}). \]

• Given this function \( V(k_1) \), the planner’s problem in period 0 would be:

\[ \max_{k_1} [\ln(Ak_0^\alpha - k_1) + \beta V(k_1)]. \]

• If the function \( V(k_1) \) known, we could solve for the policy function \( h \) such that:

\[ k_1 = h(k_0) \quad \text{and} \quad c_0 = Ak_0^\alpha - h(k_0). \]
• But note: $V(k_0)$ is the maximized value of the original problem. Thus $V(k_0)$ must be the maximized value of the one-period time-zero problem as written in previous overhead:

$$V(k_0) = \max_{k_1} [\ln(Ak_0^\alpha - k_1) + \beta V(k_1)].$$

- Now we have the problem in a recursive structure.
- If we go out t+1 periods we get

$$V(k_t) = \max_{k_{t+1}} [\ln(Ak_t^\alpha - k_{t+1}) + \beta V(k_{t+1})].$$

• Drop the subscripts:

$$V(k) = \max_{k'} [\ln(Ak^\alpha - k') + \beta V(k')],$$

• The solution to this Bellman equation is:

$$V(k) = \ln(Ak^\alpha - h(k)) + \beta V(h(k))$$

This equation is a functional equation and can be solved for the pair of unknown functions $V(k)$ and $h(k)$. 
The general deterministic control problem

\[
\max_{\{u_t\}_0^\infty} \sum_{t=0}^\infty \beta^t r(x_t, u_t)
\]

subject to

\[
x_{t+1} \in g(x_t, u_t)
\]

\[x_0 \text{ given}\]

• We let \(r(x_t, u_t)\) denote the single period return function.

• We let \(g(x_t, u_t)\) denote the set of constraints the determine the feasible choices of \(u_t\) given \(x_t\).

• \(u_t\) are called control variables.

• \(x_t\) are called state variables.
We define

\[ V^*(x_0) \equiv \max_{\{u_t\}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \text{ subject to } x_{t+1} \in g(x_t, u_t) \]

for a given \( x_0 \).

So in the general set-up we write down the Bellman equation as:

\[ V(x) = \max_u \{ r(x, u) + \beta V(x') \} \] (1)

subject to:

\[ x' \in g(x, u) \]
Properties of the Bellman Equation
Most standard discussions of dynamic programming assume the following:

1. \( r \) is concave and bounded, and
2. the set \( \{(x_{t+1}, x_t) : x_{t+1} \in g(x_t, u_t), u_t \in \mathbb{R}^k\} \) is convex and compact.

Then we can show:

1. \( V(x) \) is a monotonically increasing function (Stokey-Lucas-Prescott Theorem 4.7).
2. \( V(x) \) is strictly concave and \( h(x) \) is a continuous single-valued function (SLP Theorem 4.8).
3. (The Principle of Optimality) The solution to (1) is \( V^*(x_0) \).
4. This solution is approached in the limit as \( j \rightarrow \infty \) by iterations on:
   \[
   V_{j+1}(x) = \max_u \{ r(x, u) + \beta V_j(x') \}
   \]
   subject to
   \[
   x' \in g(x, u)
   \]
   Note we are iterating “backwards.”
5. (Benveniste and Scheinkman) The limiting value function \( V \) is differentiable with
   \[
   V'(x) = \frac{\partial r}{\partial x}[x, h(x)] + \beta \frac{\partial g}{\partial x}[x, h(x)]V'(g[x, h(x)])
   \]
   This is Theorem 4.10 in SLP.
Note that if we go back to our special case of the one sector growth model and apply Benveniste and Scheinkman, we get:

\[ V'(k) = \frac{\alpha A k^{\alpha - 1}}{A k^{\alpha} - k'}. \]

Take the FOC for the right-hand side of (1) yields:

\[ 0 = \frac{-1}{A k^{\alpha} - k'} + \beta V'(k') \]

These two derivatives imply:

\[ \frac{1}{A k^{\alpha} - k'}(-1) + \beta \frac{\alpha A k'^{\alpha - 1}}{A k'^{\alpha} - k''} = 0 \]
Solving the Bellman equation

1. Value function iteration (working backwards)
2. Guess and verify
3. Policy iteration
**Method 1: Value function iteration (working backwards)**

We are looking for a fixed point of a concave functional equation. So we can:

1. Start off with a bounded and continuous initial $V_0(x)$.
2. Solve the one period problem

$$V_1(x) = \max_u \{ r(x, u) + \beta V_0(x') \}$$

subject to

$$x' \in g(x, u)$$

3. Take the value $V_1(x)$ that solves the above maximization problem and solve

$$V_2(x) = \max_u \{ r(x, u) + \beta V_1(x') \}$$

subject to

$$x' \in g(x, u)$$

4. Repeat until the $h_j(x)$ “stops changing” and/or $V_j(x)$ and $V_{j+1}(x)$ are “close.”
1. Choose $V_0(k) = 0$.
2. Set $V_1(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta 0 \}$.
3. The solution is to set $k' = 0$ so $V_1(k^\alpha) = \ln(A + \alpha \ln k)$.
4. Set $V_2(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta (\ln A + \alpha \ln k') \}$.
5. The solution is to set $k' = \frac{\beta \alpha}{1 + \beta \alpha} A_k^\alpha$ so

$$V_2(k) = \ln \left( \frac{A k^\alpha}{1 + \beta \alpha} \right) + \beta \ln A + \frac{\beta \alpha}{1 + \beta \alpha} \ln k.$$ 

6. Repeat enough times until you see that the value function follows a geometric sequence that converges to $V(k) = 1 \left[ \ln(A(1 - \beta \alpha)) + 1 - \beta \alpha \frac{\ln(A \beta \alpha)}{1 - \alpha \beta} \right] + \frac{\alpha}{1 - \alpha \beta} \ln k$.

The associated policy function is $k' = \beta \alpha A_k^\alpha$. 
Method 2: Guess and Verify

• Guess a functional form for $V(k)$.

• Verify the functional form of type guessed solves the Bellman equation and deduce the values for the coefficients.

There are only a handful of models for which you can use this method. How do you form a good guess?
Back to the OSGM II

1. Guess $V(k) = E + F \ln(k)$.

2. Now verify

$$E + F \ln(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta(E + F \ln(k')) \}$$

- Take the first-order condition

$$0 = -\frac{1}{Ak^\alpha - k'} + \beta F \frac{1}{k'}$$

Solving for $k'$

$$k' = \frac{\beta F}{1 + \beta F} Ak^\alpha$$

- Substitute back in to the Bellman equation

$$E + \ln k = \ln \left( Ak^\alpha - \frac{\beta F}{1 + \beta F} Ak^\alpha \right) + \beta \left( E + F \ln \left( \frac{\beta F}{1 + \beta F} Ak^\alpha \right) \right)$$

$$= \ln \left( \frac{A}{1 + \beta F} \right) + \alpha \ln k + \beta E + \beta F \ln \left( \frac{\beta F A}{1 + \beta F} \right) + \beta F \alpha \ln k.$$
• Equate coefficients on the \( \ln k \) terms

\[
F \ln k = \alpha \ln k + \beta F \alpha \ln k
\]

\[
F = \frac{\alpha}{1 - \beta \alpha}
\]  

(2)

• Equate coefficients on the constant term

\[
E = \ln \left( \frac{A}{1 + \beta F} \right) + \beta E + \beta F \ln \left( \frac{\beta AF}{1 + \beta F} \right)
\]

• Use equation (2) to substitute out \( F \)

• Do a lot of algebra, and solve for \( E \)
Method 3: Policy Function Iteration

1. Pick a feasible policy, \( u = h_0(x) \), and compute the value of sticking with that policy forever:
   \[
   V_{h_0}(x) = \sum_{t=0}^{\infty} \beta^t r(x_t, h_0(x_t)), \quad \text{where} \quad x_{t+1} = g(x_t, h_0(x_t)).
   \]

2. Choose a new policy function \( h_1(x) \) that maximizes the following two period problem:
   \[
   \max_u \{ r(x, u) + \beta V_{h_0}(g(x, h_0(x))) \}
   \]

3. Repeat steps 1 and 2 with the updated policy function until the policy function “stops changing.”
1. Pick feasible policy function $k_{t+1} = h_0(x) = \frac{1}{2} A k_t^\alpha$

2. Compute

$$V_{h_0}(x) = \sum_{t=0}^{\infty} \beta^t \ln \left( A k_t^\alpha - \frac{1}{2} A k_t^\alpha \right)$$

$$= \sum_{t=0}^{\infty} \beta^t \ln \left( \frac{1}{2} A k_t^\alpha \right)$$

$$= \sum_{t=0}^{\infty} \beta^t \left( \ln \left( \frac{1}{2} A \right) + \alpha \ln k_t \right)$$

Note

$$k_t = \frac{1}{2} A k_{t-1}^\alpha$$

$$= \frac{1}{2} A \left[ \frac{1}{2} A k_{t-2}^\alpha \right]^\alpha$$

$$= \left( \frac{1}{2} \right)^{1+\alpha} A^{1+\alpha} k_{t-2}^{\alpha^2}$$

repeated recursive substitution yields:

$$k_t = D k_0^{\alpha^t}$$

where $D$ denotes a constant term so,

$$\ln k_t = \ln D + \alpha^t k_0.$$
Therefore
\[
V_{h_0}(x) = \sum_{t=0}^{\infty} \beta^t \left( \ln \left( \frac{1}{2} A \right) + \alpha \ln D + \alpha^{t+1} \ln k_0 \right)
\]
\[
= \text{constant term} + \frac{\alpha}{1 - \beta \alpha} \ln k_0
\]

3. Evaluate the two period problem:
\[
\max_{k'} \left\{ \ln (A k'^\alpha - k') + \beta \left[ \text{constant term} + \frac{\alpha}{1 - \beta \alpha} \ln k' \right] \right\}
\]

Taking the first-order condition yields:
\[
\frac{-1}{A k'^\alpha - k'} + \frac{\beta \alpha}{1 - \beta \alpha} \frac{1}{k'} = 0.
\]

Thus
\[
k' = \alpha \beta A k'^\alpha.
\]

*The Howard improvement algorithm converges in a single step!*