1. Consider the following problem in which a representative agent is endowed with one unit of time each period and initial capital stock $k_0$ at time zero. Each period the agent chooses consumption ($c_t$), investment ($k_{t+1}$), and labor ($n_t$) to maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln c_t + \gamma \ln (1 - n_t)]$$

subject to

$$c_t + k_{t+1} = A_t k_t^\alpha n_t^{1-\alpha}$$

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1}$$

where $\gamma$ is a positive parameter, $\rho$ is a parameter that lies in the interval $(-1, 1)$ and $\epsilon_{t+1}$ represents a white noise error process. Assume $k_{t+1}$, $n_t$, and $c_t$ are chosen after $\epsilon_t$ is observed but before $\epsilon_{t+1}$ is observed.

(a) Formulate the Bellman equation for this problem. Identify the state and control variables.

(b) Verify that the value function has the form

$$V(k, A) = D + G \ln k + H \ln A.$$ 

Solve for the coefficients $G$ and $H$. Solve for the decision rules for $k_{t+1}$, $c_t$ and $n_t$. Do all three decision rules depend on the states? Explain briefly.

(c) Three well-known stylized facts about macro-time series are

i. Output, consumption, and investment data exhibit strong persistence over time.

ii. Investment is more volatile than output. Consumption is less volatile than output.

iii. Employment is strongly pro-cyclical (i.e. it moves with output) and percentage fluctuations in employment are larger than in output.

How well does this model match these stylized facts? You can answer this question via “pencil and paper” solutions to the model. If you want, you can verify your answers on the computer.
2. Consider a version of the stochastic growth model in which a representative agent chooses consumption \((c_t)\) and investment \((i_t)\) each period to maximize:

\[
E_0 \sum_{t=0}^{\infty} \left\{ \beta \frac{1}{1-\sigma} (c_t^{1-\sigma} - 1) \right\}.
\]

The representative agent has access to a production function \(y_t = z_t k_t^\alpha\), where \(z_t\) is a random productivity shock. Assume \(z_t\) can take on one of two values: high or low. Further assume \(z_t\) evolves according to a two state Markov chain with a transition matrix \(\chi\) where:

\[\chi(z, z') = \text{Prob}(z_{t+1} = z' | z_t = z) \quad \text{for} \quad z, z' \in \{\text{high}, \text{low}\}.\]

Output can be used for consumption or investment each period:

\[c_t + i_t = y_t.\]

Capital partially depreciates each period, so its law of motion is:

\[k_{t+1} = (1-\delta)k_t + i_t.\]

If I were taking this course, I would modify the Matlab code posted on the course web page to answer this problem. Set \(\sigma = 2, \alpha = 0.40, 1-\delta = 0.90, \beta = 0.98, z_{\text{high}} = 1.5\) and \(z_{\text{low}} = 0.5\) with

\[\chi(z, z') = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}.\]

You may also need to modify the capital grid so that model’s solution does not depend on your choice of the upper bound of capital.

(a) Solve the model numerically and compute the means and standard deviations for the steady-state distributions of output, consumption, investment, and capital.

(b) Compare this economy to an economy in which the shocks to \(z\) are eliminated. That is, we want to compare two economies: one which the variance of \(z\) is positive, and one in which the variance of \(z\) is zero. For the deterministic economy, set the level of \(z\) equal to the mean of \(z\) from the stochastic economy.

How does the average level of the capital stock and thus the average level of output vary across the two economies? In which economy is average consumption higher? Provide the economic intuition for this result.

Suppose an agent can choose which economy (the stochastic or the deterministic) he/she would like to live. Which economy would the agent choose? \[\text{[Hint: What is the economic interpretation of the value function?]}\]