I. Recap from deterministic dynamic programming

2. The stochastic control problem

3. A simple stochastic growth model

4. Steady-state distributions

Stochastic Dynamic Programming
Recap

Recall the general form of the deterministic control problem:

$$\max_{\{u_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

subject to

$$x_{t+1} \in g(x_t, u_t)$$

$$x_0 \text{ given}$$

where

- $r(x_t, u_t)$ is the single period return function.
- $g(x_t, u_t)$ is the set of constraints the determine the feasible choices of $u_t$ given $x_t$.
- $u_t$ are the control or choice variables.
- $x_t$ are the state variables.
Modify problem to permit uncertainty of particular kinds.

\[
\max_{\{u_t\}} \mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)
\]  

subject to

\[
x_{t+1} \in g(x_t, u_t, \epsilon_{t+1})
\]

\[x_0 \text{ given}\]

- We let \( \epsilon_t \) be a sequence of independently and identically distributed random variables.
- We assume that at time \( t \), the current and past history of the state variables \( \{x_t, x_{t-1}, ..., x_0\} \) are known. We assume all \( x_{t+j} \) for \( j \geq 1 \) are unknown at date \( t \).
- The operator \( \mathcal{E}_t \) takes the expectations of a random variable given the information known at time \( t \).
- Finally we assume the shock \( \epsilon_{t+1} \) is realized after the control variable \( u_t \) has been chosen.
Properties of the Bellman Equation

We can write the Bellman equation for the stochastic control as:

\[ V(x) = \max_u \{ r(x, u) + \beta E[V(x')|x] \} \]  

subject to

\[ x' \in g(x, u, \epsilon') \]

If the one-period return function is bounded, and the constraint set is convex and compact, then:

1. \( V(x) \) is a monotonically increasing function
2. \( V(x) \) is strictly concave and \( h(x) \) is a continuous single-valued function
3. (The Principle of Optimality) The solution to the Bellman equation is \( V^*(x_0) \) where

\[ V^*(x_0) \equiv \max_{\{u_t\}_0} E \sum_{t=0}^\infty \beta^t r(x_t, u_t) \text{ subject to } x_{t+1} \in g(x_t, u_t, \epsilon_{t+1}) \]

for a given \( x_0 \).
4. This solution is approached in the limit as \( j \to \infty \) by iterations on:

\[ V_{j+1}(x) = \max_u \{ r(x, u) + \beta E[V_j(x')|x] \} \]

subject to

\[ x' \in g(x, u, \epsilon') \]
5. The limiting value function \( V \) is differentiable.
A Simple Stochastic Growth Model

\[
\max_{\{c_t,k_{t+1}\}_0} \sum_{t=0}^{\infty} \beta^t \ln(c_t)
\]

subject to:

\[
c_t + k_{t+1} \leq A_t k_t^\alpha
\]

\[
k_{t+1} \geq 0
\]

and we are going to let \(\{A_t\}\) be a sequence of i.i.d. random variables.

So what are the objects of choice in this model? They are set of contingency plans:

\[
c_0, c_1(A_1), c_2(A_1, A_2), ..., c_t(A_1, A_2, ..., A_t), ...
\]

\[
k_1, k_2(A_1), k_2(A_1, A_2), ..., k_{t+1}(A_1, A_2, ..., A_t), ...
\]
Assume $A_t$ can take on two values each period: high or low.

\[
E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) = \sum_{t=0}^{\infty} \beta^t E_0 \ln(c_t)
\]
\[
= \sum_{t=0}^{\infty} \beta^t E_0 \ln(c_t(A_1, A_2, \ldots, A_t))
\]

So at period 1

\[
E_0 \ln(c_1) = \text{prob}(A_1 = \text{high}) \ln(c_1(A_{\text{high}})) + \text{prob}(A_1 = \text{low}) \ln(c_1(A_{\text{low}}))
\]

In period 2

\[
E_0 \ln(c_2) = (\text{prob}(A_1 = \text{high}) \times \text{prob}(A_2 = \text{high})) \ln(c_2(A_{\text{high}}, A_{\text{high}})) + \\
(\text{prob}(A_1 = \text{high}) \times \text{prob}(A_2 = \text{low})) \ln(c_2(A_{\text{high}}, A_{\text{low}})) + \\
(\text{prob}(A_1 = \text{low}) \times \text{prob}(A_2 = \text{high})) \ln(c_2(A_{\text{low}}, A_{\text{high}})) + \\
(\text{prob}(A_1 = \text{low}) \times \text{prob}(A_2 = \text{low})) \ln(c_2(A_{\text{low}}, A_{\text{low}}))
\]

So at date $t$ there are $2^t$ possible shock paths.
We write the Bellman equation as:

\[ V(k, A) = \max_{c, k'} \{ \ln c + \beta E[V(k', A')] \} \]

subject to

\[ c + k' \leq Ak^\alpha \]
\[ k' \geq 0 \]
\[ A' \text{ is i.i.d.} \]

We assume \( A \) takes on \( n \) values \( A^1, A^2, \ldots, A^n \) with probabilities \( \pi_1, \pi_2, \ldots, \pi_n \). We place the usual restriction on \( \pi \): \( \pi_i \geq 0 \ \forall i \) and \( \sum_i \pi_i = 1 \).

\[ E[V(k', A')] = \sum_{j=1}^{n} \pi_j V(k', A'^j) \]
If we substitute the constraint into the objective, we can write the Bellman equation as:

\[
V(k, A) = \max_{k'} \{ \ln (Ak^\alpha - k') + \beta E[V(k', A')] \}
\]

Taking the first-order condition yields:

\[
0 = \frac{-1}{Ak^\alpha - k'} + \beta E[V_1(k', A')]
\]

Applying Benvensite-Scheinkman (the envelop condition) yields:

\[
V_1(k, A) = \frac{\alpha A k^{\alpha - 1}}{Ak^\alpha - k'}
\]

So we get

\[
0 = \frac{-1}{Ak^\alpha - k'} + \beta E \left[ \frac{\alpha A' k'^{\alpha - 1}}{A' k'^{\alpha - 1} - k''} \right]
\]
We can guess and verify that the solution to this problem is of the form:

\[ V(k, A) = F + G \ln k + H \ln A \]

and the optimal policy rule is:

\[ k' = \alpha \beta A k^\alpha \]

So \( k' = h(k, A) \) is a stochastic difference equation, so capital follows a Markov process.

We know in the deterministic case we solve for a steady state of capital:

\[ k' = \alpha \beta A k^\alpha \]

Taking logs:

\[ \ln k' = \ln(\alpha \beta A) + \alpha \ln k \]

\[ \ln(k_{ss}) = \frac{\ln(\alpha \beta A)}{1 - \alpha} \]
We can say some things about the stationary distribution of capital:

\[ k' = \alpha \beta A k^\alpha \]

Taking logs:

\[ \ln k' = \ln(\alpha \beta) + \ln(A) + \alpha \ln k \]

Assume \( \ln(A_t) \) is an i.i.d. random variable whose \( E[\ln A_t] = 0 \), and \( \text{var}[\ln A_t] = E[(\ln A_t - E[\ln A_t])^2] = \sigma_A^2 \).

Then we get

\[ E[\ln k_{t+1}] = E[\ln(\alpha \beta) + \ln A + \alpha \ln k_t] \]

\[ E[\ln k_{t+1}] = \ln(\alpha \beta) + \alpha E[\ln k_t] \]

To compute the variance of the distribution of capital:

\[ \text{var}[\ln k_{t+1}] = \text{var}[\alpha \ln(k_t) + \ln A_t] \]

\[ = \text{var}(\alpha \ln(k_t)) + \text{var}(\ln A_t) \]

\[ = \alpha^2 \text{var}(\ln k_t) + \sigma_A^2 \]

So

\[ \text{var}(\ln k_t) \rightarrow \frac{\sigma_A^2}{1 - \alpha^2} \text{ as } t \rightarrow \infty \]