Entanglement Entropy, Quantum Field Theory, and Holography

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Contents

0  Intro & disclaimer  2 

1  

2  

1
0 Intro & disclaimer

Over past 10 years, explosion of activity in entanglement entropy in QFT:
- many conceptual ramifications
- enormous number of applications.

It would be impossible for me to cover even just the most important developments in one hour. I will focus on a small & idiosyncratic selection of topics, emphasizing some important open problems.

Citations will generally be limited to the paper that (as far as I know) initiated the given subject.

1 Basic definitions

Divide a quantum system into subsystems $A, A^c$, such that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$.

Entangled pure state: $|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_{A^c}$.

Entanglement leads to mixedness of reduced density matrix:

$$\rho_A := \text{tr}_{\mathcal{H}_{A^c}} |\psi\rangle \langle \psi| = \sum_i |\lambda_i|^2 |i\rangle_A \langle i|_A$$
Entanglement is quantified by entropy of $\rho_A$—**entanglement entropy**:

$$S(A) := - \text{tr} \rho_A \ln \rho_A = - \sum_i |\lambda_i|^2 \ln |\lambda_i|^2$$

Entanglement is a ubiquitous phenomenon in quantum systems. EE is a central concept in quantum statistical mechanics & quantum information theory.

For a mixed state $\rho$, we similarly define

$$\rho_A := \text{tr}_{\bar{A}} \rho, \quad S(A) := - \text{tr} \rho_A \ln \rho_A.$$  

Also called EE (although it doesn’t measure just entanglement).

## 2 Entanglement entropy in quantum field theories

Consider a lattice system in $D - 1$ spatial dimensions with Hilbert space $\mathcal{H} = \bigotimes_{\text{sites}} \mathcal{H}_i$, local Hamiltonian.

Let $A$ be a region of size $L \gg \epsilon = $ lattice spacing. Two patterns are observed:

- In a generic state,
  
  $$S(A) \sim \ln \dim \mathcal{H}_A \sim (\# \text{ lattice sites in } A) \sim (L/\epsilon)^{D-1}$$

- In the ground state (or other low-lying pure state),
  
  $$S(A) \sim (\# \text{ links cut by } \partial A) \sim (L/\epsilon)^{D-2} \quad (\text{or } \sim \ln \epsilon \text{ in } D = 2)$$  

  \begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure}
  \caption{Diagram of a lattice system with a region $A$.}
  \end{figure}

  (Bombelli, Koul, Lee, Sorkin ’86, . . .)

Typical physical states have very little entanglement, & most of it is local. (Basis for efficient numerical simulation methods: build lack of entanglement into variational ansatz—DMRG, MPS, PEPS, MERA, etc.; see review Vidal ’09.)

In $\epsilon \to 0$ limit, system may be describable by a QFT. Holding $A$ fixed, $S(A) \to \infty$. By removing UV-divergent part, can we extract “universal” quantities that characterize the QFT (are independent of lattice realization)?

Following some early work (Bombelli, Koul, Lee, Sorkin ’86, Srednicki ’93, Callan & Wilczek ’94, Holzhey, Larsen, Wilczek ’94, . . .), starting ~ 10 years ago, 3 key advances convinced people that the answer is yes, and that EE is a useful tool for studying QFTs:

- **Calabrese, Cardy ’04**: In $D = 2$ critical models, EEs are related to twist-field correlation functions in cyclic orbifold CFTs; general formula for ground-state EE of an interval:
\[
S(A) = \frac{c}{3} \ln \frac{L}{\epsilon} + \text{(non-universal constant)}
\]

Application: diagnosing criticality and determining \(c\) from lattice simulations.

- **Kitaev, Preskill '05, Levin, Wen '05**: In gapped \(D = 3\) theories, for a simply-connected region much larger than correlation length,

\[
S(A) = \frac{L}{\epsilon} \times \text{(non-universal constant)} - \gamma,
\]

where topological EE \(\gamma\) characterizes topological QFT that controls IR.

Application: diagnosing topological order & phase transitions.

- **Ryu, Takayanagi '06**: In holographic CFT, conjecture for EE of arbitrary region:

\[
S(A) = \frac{1}{4G_N} \text{area}(m(A))
\]

\[
m(A) = \text{minimal surface in bulk anchored on } \partial A
\]

UV divergence arises from part of \(m(A)\) near boundary.

Application: many (see below).

Examples of quantities that are (believed to be) UV-finite & universal, characterize physics at scale \(L\):

- **Renormalized EE**:

\[
D = 2 : \quad F(L) = L \frac{d}{dL} S(A_L)
\]

\[
D = 3 : \quad F(L) = \left( \frac{d}{dL} - 1 \right) S(A_L)
\]

\[
D = 4 : \quad F(L) = \frac{1}{2} L \frac{d}{dL} \left( \frac{d}{dL} - 2 \right) S(A_L)
\]

\[
\vdots
\]

where \(A_L\) is a family of regions related by uniform dilatation.

(Casini, Huerta '04, . . . , Liu, Mezei '12, . . .

- **Mutual information**:

\[
I(A : B) := S(A) + S(B) - S(AB)
\]

where \(A, B\) do not share a boundary.

Measures entanglement + classical correlation.

(Calabrese, Cardy '04, . . .

- **Tripartite information**:

\[
\text{.}
\]

\[
\text{.}
\]
\[ I_3(A:B:C) := S(ABC) + S(A) + S(B) + S(C) - S(AB) - S(AC) - S(BC) \]
\[ = I(A:B) + I(A:C) - I(A:BC) \]

Measures correlations of correlations.  
(Kitaev, Preskill ’05, . . .)

Since 2004, an explosion of activity in the study of EEs & related quantities in QFTs, addressing old problems & posing new ones.

Examples (almost all are studied in both holographic & non-holographic theories):

- Dependence of \( S(A) \) on state, as well as on geometry & topology of \( A \) (e.g. divergences from singularities in geometry of \( \partial A \)).
- Characterizing fixed points & constraining RG flows: C-theorem in \( D = 2 \), F-theorem in \( D = 3 \), etc. (Casini, Huerta ’04, Myers, Sinha ’10, Casini, Huerta ’12, . . .).
- Probe of confinement (Kutasov, Klebanov, Murugan ’07, Velytsky ’08, . . .).
- Condensed-matter applications (e.g. probe of Fermi-liquid vs. non-Fermi-liquid behavior).
- Probe of quenches & thermalization processes, “propagation” of entanglement (Calabrese, Cardy ’05, Abajo-Arrastia, Aparicio, López ’10, . . .).
- Probe of correlations of fields on cosmological backgrounds (Maldacena, Pimentel ’12, . . .).
- Bekenstein-Hawking entropy as EE of fields on black-hole background: \( S_{\text{BH}} \sim S(\text{exterior}) \) (Bombelli, Koul, Lee, Sorkin ’86, . . .).
- Related information-theoretic quantities, such as entanglement negativities (Calabrese, Cardy, Tonni ’13, . . .), relative entropies (Blanco, Casini, Hung, Myers ’13, Lashkari ’14), etc.
- Structure of reduced density matrix \( \rho_A \) (Casini, Huerta ’09, Hung, Myers, Smolkin, Yale ’11, . . .).

There remain profound problems of practice & principle in definition & calculation of EEs in QFTs, e.g.:

- The main tool for calculating EEs analytically in (non-holographic) QFTs is the replica trick (Holzhey, Larsen, Wilczek ’94, . . .):
  1. calculate entanglement Rényi entropy (ERE)
  \[ S_{\alpha}(A) := \frac{1}{1 - \alpha} \ln \text{tr} \rho_A^\alpha \]
  for \( \alpha = 2, 3, . . . \) in terms of a Euclidean path integral on a certain branched-cover manifold
  2. fit to analytic function of \( \alpha \)
  3. evaluate at \( \alpha = 1 \): \( \lim_{\alpha \to 1} S_{\alpha}(A) = S(A) \).
Unfortunately, this method is roundabout & often impossible to carry out. E.g. in free compact boson CFT (Luttinger liquid), for 2 separated intervals, $S_\alpha(AB)$ is known explicitly for $\alpha = 2, 3, \ldots$ (Calabrese, Cardy, Tonni '09), but it is not known how to fit the values to an analytic function, so EE (and mutual information) remains unknown.

Challenge 1: Find a more direct way to analytically calculate EEs.

- It is awkward that to define EE one has to fall back on a lattice regularization (e.g. breaks Lorentz invariance). Also, in (lattice or continuum) theory with (discrete or continuous) gauge symmetry, $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_{A'}$, because of Gauss-law constraint (Buividovich, Polikarpov '08, ...); to define EE one must enlarge the Hilbert space in an ad hoc way.

The path integral used to calculate $S_\alpha(A)$ ($\alpha = 2, 3, \ldots$) can be regulated in any convenient way & automatically takes proper account of gauge symmetries (Headrick, Lawrence, Roberts '12, ...). In principle this can be used to define EE. However, this obscures the special properties of EE.

Challenge 2: Give a good general field-theoretic definition of EE (or related universal quantities).

Hilbert space of a QFT can also be factorized in other ways (e.g., given a perturbative realization, by regions of momentum space: Balasubramanian, McDermott, Van Raamsdonk '11, ...). Is it possible to extract universal quantities from such EEs?

3 Lightning review of holographic dualities (a.k.a. AdS/CFT correspondence)

Certain $D$-dimensional CFTs are equivalent to $D+1$ dimensional quantum gravity theories (QGTs).

QGT has negative cosmological constant ($\Lambda = -R^2_{\text{AdS}}$) & is subject to asymptotically anti-de Sitter (AdS) boundary conditions. An asymptotically AdS spacetime has a spatial “boundary” that is infinitely far away but, due to warping, can be reached by a light ray in finite time; an observer can see the boundary, which has a reflecting boundary condition. This boundary can be identified with the spacetime where the CFT lives.

If CFT has $N$ fields (e.g. $N \sim n^2$ for $U(n)$ gauge theory), then

$$\frac{R^{D-1}_{\text{AdS}}}{G_N \hbar} = \left( \frac{R_{\text{AdS}}}{l_{\text{Planck}}} \right)^{D-1} \sim N.$$ 

QGT becomes classical in $N \to \infty$ limit. Classical gravity is a collective or thermodynamic description of the $N$ fields.
Each solution of gravity field equations represents a state (pure or mixed) of CFT. E.g.:
- AdS represents vacuum
- Black hole represents deconfined thermal state. Bekenstein-Hawking entropy formula:
  \[ S_{\text{BH}} = \frac{1}{4G_N} \text{area(horizon)} = O(N) \]

4 Ryu-Takayanagi formula

Consider a static solution. Constant-time slice is an asymptotically hyperbolic space.

Ryu-Takayanagi formula:

\[ S(A) = \frac{1}{4G_N} \text{area}(m(A)) \]
\[ m(A) = \text{minimal bulk surface homologous to } A \]

Large amount of evidence in favor of RT formula:
- has been applied to a wide variety of holographic systems, always giving physically reasonable results
- agrees with EEs computed from first principles (usually using replica trick) in many specific cases (both divergent & finite parts); general argument (Lewkowycz, Maldacena '13)
- obeys all applicable properties of EEs, e.g.
  1. if full system is in a pure state, then \( S(A) = S(A^c) \); if full system is mixed, then not necessarily
  2. strong subadditivity: \( S(AB) + S(BC) \geq S(B) + S(ABC) \) (Headrick, Takayanagi '07, . . .)
RT formula implies special properties of EEs in holographic theories, e.g.

- competing minimal surfaces ⇒ EE has phase transitions under continuous deformations of region, e.g.

\[ m(AB) = m(A) \cup m(B) \]

- monogamy of mutual information:

\[ I_3(A : B : C) \leq 0, \quad \text{i.e.} \quad I(A : BC) \geq I(A : B) + I(A : C) \]

Proof is similar to SSA proof (Hayden, Headrick, Maloney ’11, …)

Effects of higher-derivative & perturbative quantum \((1/N)\) corrections in bulk have been investigated (Fursaev ’06, Faulkner, Lewkowycz, Maldacena ’13, …).

There are also important non-perturbative quantum \((1/N)\) corrections, e.g. to smooth out phase transitions, which are not understood.

Challenge 3: Understand non-perturbative quantum corrections to RT formula.

EREs have also been calculated in holographic theories (Headrick ’10, …). Involves finding bulk solutions with complicated boundary conditions (vs. applying RT to given solution for EE). This is related to the fact that Rényi entropies are not good thermodynamic quantities.

Challenge 4: Understand the structure of \(\rho_A\) in holographic theories.

Evidence from several directions suggests that a large class of large-\(N\) theories shares special properties of holographic EEs, e.g.

- \(S(A)\) given by minimization
- phase transitions in \(S(A)\) as \(A\) is varied
- strict vanishing (at order \(N^0\)) of \(I(A : B)\) when \(A, B\) are sufficiently far apart
- generic saturation (at order \(N^0\)) of Araki-Lieb inequality: \(S(AB) = S(A) - S(B)\)
- monogamy of mutual information

(Headrick ’10, …).

Challenge 5: Understand EEs and reduced density matrices in general large-\(N\) theories.
5 Hubeny-Rangamani-Takayanagi formula

To avoid RT formula’s static condition, Hubeny, Rangamani, Takayanagi (’07) proposed a covariant generalization:

\[
S(A) = \frac{1}{4G_N} \text{area}(m(A)) \\
\text{m}(A) = \text{extremal bulk surface spacelike-homologous to } A
\]

Reduces to RT in static case.

HRT formula has been applied in many cases, always giving physically reasonable results. There is evidence that it obeys strong subadditivity, when null energy condition is obeyed (Allais, Tonni ’11, . . . ). Although it has apparently acausal behavior (e.g. it can “see” behind the horizon of a non-static black hole), it is consistent with causality in the CFT (Wall ’12; Headrick, Hubeny, Lawrence, Rangamani ’14):

Given \( A \), we can decompose boundary into 4 spacetime regions:

\[
D(A), \quad D(A^c), \quad J^+(\partial A), \quad J^-(\partial A)
\]

Only perturbations in \( J^-(\partial A) \) should be able to change \( S(A) \). A field-theory perturbation sources a bulk perturbation that propagates in from boundary. If it reaches \( m(A) \), it can change its area. So, for consistency of HRT, \( m(A) \) should be reachable only by perturbations in \( J^-(\partial A) \). Theorem: If bulk obeys NEC, then

\[
J^\text{bulk}_{m(A)} \cap \text{bdy} = J^-(\partial A)
\]
For example, suppose bulk is a two-sided black hole. This represents two entangled but non-interacting CFTs. If $A$ is (a Cauchy slice for) all of $\text{bdy}_1$, theorem prevents $S(A)$ from being influenced by perturbations on either boundary. Hence $m(A)$ lies inside “causal shadow”: set of points not causally connected to either boundary.

There remain many fundamental questions about the HRT proposal. **Challenge 6:** Show that, under physically reasonable assumptions, HRT formula has (or doesn’t have) the following properties:

- $m(A)$ exists
- Strong subadditivity is obeyed
- $S(A)$ is continuous under continuous variations in $A$
- Monogamy of mutual information is obeyed
- It can be derived!

By reformulating HRT in terms of a maximin construction, Wall (’12) has made significant progress on several of these problems.