(1) Hatcher 3.3.7
(2) (Proving Poincare Duality for $\mathbb{R}^n$.) $\mathbb{R}^n$ is a noncompact orientable n-manifold.
Let $B_r \subset \mathbb{R}^n$ denote the closed metric ball of radius $r$ centered at the origin.
(a) Prove that for each $r \in \mathbb{N}$, $H^k(\mathbb{R}^n, \mathbb{R}^n - B_r) = \mathbb{Z}$.
(b) Prove that the usual cap product has a well defined relative version $H_i(X, A) \cap H^k(X, A) \rightarrow H_{i-k}(X)$,
(c) Let $\mu_r$ denote a generator of $H^k(\mathbb{R}^n, \mathbb{R}^n - B_r) = \mathbb{Z}$ from part (a). Then for each $r \in \mathbb{N}$ consider the relative poincare dual homomorphism map $PD^k_r: H^k(\mathbb{R}^n, \mathbb{R}^n - B_r) \rightarrow H_{n-k}(\mathbb{R}^n)$ given by $\alpha \mapsto [\mu_r] \cap \alpha$. Then define $PD^k: \lim_{\rightarrow} H^k(\mathbb{R}^n, \mathbb{R}^n - B_r) = H^k(\mathbb{R}^n) \rightarrow H_{n-k}(\mathbb{R}^n)$ as the direct limit of the homomorphisms $PD^k_r$. Prove that $PD^k$ is an isomorphism for all $k$.
(3) Let $M$ be a closed, oriented 4n-dimensional manifold, with fundamental class $[M]$. Define an intersection pairing
\[<, >: H^{2n}(M, \mathbb{Z})/Tor \times H^{2n}(M, \mathbb{Z})/Tor \rightarrow \mathbb{Z}\]
given by
\[< \alpha, \beta > \mapsto (\alpha \cup \beta) \cap [M].\]
(a) Let $\alpha_i$ be a $\mathbb{Z}$-basis of $H^{2n}(M, \mathbb{Z})/Tor$, and define the matrix
\[A = (a_{i,j}) = (< \alpha_i, \alpha_j >).\]
Prove that the matrix $A$ is symmetric and hence diagonalizable over $\mathbb{R}$. Moreover, prove that $0$ is not an eigenvalue of $A$. Then define the signature of $M$ as follows:
\[\sigma(M) := |\text{positive eigenvalues}| - |\text{negative eigenvalues}|.\]
(b) Compute $\sigma(\mathbb{C}P^2), \sigma(\mathbb{C}P^2 \# \mathbb{C}P^2), \sigma(S^2 \times S^2)$.
Where $\mathbb{C}P^2 \# \mathbb{C}P^2$ is the connect sum of two copies of $\mathbb{C}P^2$. (See the final from last semester for more information on the connect sum)
(c) Prove that $\sigma(M) = \chi(M)(mod2).$