Definition 1.10. If $S$ is a subset of a group $G$ then the subgroup generated by $S$, written $\langle S \rangle$, is defined to be the intersection of all subgroups of $G$ which contain $S$:
\[
\langle S \rangle := \bigcap_{S \subseteq H \subseteq G} H.
\]

Definition 1.11. The commutator subgroup (also called the derived subgroup) $G' = [G, G]$ is the subgroup of $G$ generated by all commutators $[a, b]$.

In general, if $A, B \leq G$ then $[A, B]$ is defined to be the subgroup of $G$ generated by all commutators $[a, b]$ where $a \in A$ and $b \in B$.


This follows immediately from the following two facts.

Lemma 1.13. \begin{enumerate}
  \item $x \langle S \rangle x^{-1} = \langle S \rangle$
  \item $x[a, b]x^{-1} = [xax^{-1}, xbx^{-1}]$.
\end{enumerate}

This lemma has a generalization:

Lemma 1.14. Given a homomorphism $\phi : G \to H, a, b \in G, S \subseteq G$ we have:
\begin{enumerate}
  \item $\phi \langle S \rangle = \langle \phi(S) \rangle$.
  \item $\phi[a, b] = [\phi(a), \phi(b)]$.
\end{enumerate}

Why is this a generalization of the previous lemma?

The main theorem about the commutator subgroup is the following.

Theorem 1.15. The image $\phi(G)$ of a homomorphism $\phi : G \to H$ is abelian if and only if the kernel of $\phi$ contains the commutator subgroup $G'$.

For this we need the following lemma whose proof is obvious.

Lemma 1.16. $S$ is a subset of $H \leq G$ iff $\langle S \rangle$ is a subgroup of $H$.

Proof of Theorem 1.15. The argument which we did in class is reversible, i.e., “iff” at every step: For any $a, b \in G$ we have
\[
[a, b] = [\phi(a), \phi(b)] = \phi[a, b].
\]

$\phi(G)$ is abelian iff the LHS is always $e$. But, the RHS is always equal to $e$ iff $[a, b] \in \ker \phi$ for all $a, b \in G$ which, by Lemma 1.16, is equivalent to saying that $G' \leq \ker \phi$.

The theorem has the following variation as an obvious corollary:

Corollary 1.17. Suppose that $N \trianglelefteq G$. Then $G/N$ is abelian iff $G' \leq N$. 

2. Solvable groups

After doing a review of group theory at lightning speed we managed to get to the first topic of the course on the first day: Solvable groups.

2.1. two definitions. I gave two equivalent definitions of a solvable group. Here is the first definition.

Definition 2.1. A group $G$ is solvable if an iterated derived subgroup $G^{(n)}$ is trivial for some positive integer $n$. Here $G^{(n)}$ is defined recursively as follows.

1. $G^{(0)} = G$
2. $G^{(i+1)} = (G^{(i)})^\prime = [G^{(i)}, G^{(i)}].$

Serge Lang uses normal towers to define solvable groups.

Definition 2.2. A normal tower for a group $G$ is a sequence of subgroups:

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\}$$

so that each subgroup is normal in the previous one: $G_i \trianglelefteq G_{i-1}$. The quotient groups $G_{i-1}/G_i$ are called the subquotients of the tower.

In general a subquotient of a group $G$ is a quotient of a subgroup of $G$. (This is more general than a subgroup of a quotient. Why is that?) Here is Lang’s definition of a solvable group.

Definition 2.3. A group is solvable if it has a normal tower whose subquotients are all abelian. Lang calls these abelian towers.

We need to show that these definitions are equivalent. One direction is obvious. The first definition implies the second. This is because the derived series:

$$G \triangleright G^\prime \triangleright G^\prime \prime \triangleright \cdots \triangleright G^{(n)} = \{e\}$$

is a normal tower with abelian subquotients. To prove the converse, we need one more corollary or Theorem 1.15.

Corollary 2.4. Suppose that $H \leq G, N \trianglelefteq G$ and $G/N$ is abelian. Then $H^\prime \leq H \cap N$.

Proof. Take the composition

$$\phi : H \hookrightarrow G \rightarrow G/N.$$ 

Since $\phi(H) \leq G/N$ is abelian,

$$H^\prime \leq \ker \phi = \{x \in H \mid \phi(x) = e\} = H \cap N.$$

$\square$
Coming back to the equivalence of definitions, suppose that $G$ has a normal tower with abelian subquotients. Since $G/G_1$ is abelian, $G' \leq G_1$. Suppose by induction that $G^{(k)} \leq G_k$. We know that $G_{k+1} \leq G_k$ with abelian quotient. The corollary tells us that
\[
(G^{(k)})' = G^{(k+1)} \leq G^{(k)} \cap G_{k+1} \leq G_{k+1}.
\]
Therefore $G^{(n)} \leq G_k = \{e\}$ making $G$ solvable by the first definition.

2.2. **degree of solvability and examples.** We say that $G$ is solvable of degree $n$ if $G^{(n)} = \{e\}$ and $G^{(n-1)}$ is nontrivial.

(1) Abelian groups are solvable of degree 1 (except for the trivial group which is solvable of degree 0).

(2) $S_3$, the symmetric group on 3 letters is solvable of degree 2.

(3) $T(n,\mathbb{Z})$, the group of unipotent matrices with coefficients in $\mathbb{Z}$ is solvable, but of what degree?

To prove that $S_3$ is solvable, take the normal tower:
\[
S_3 \supseteq A_3 \supseteq \{e\}.
\]
Here $A_3 = \{e, (123), (132)\}$ is the alternating group. This is a cyclic group and thus abelian and $S_3/A_3 \cong \mathbb{Z}/2$ is also abelian. So, $S_3$ is solvable of degree 2.

As I mentioned in class, **unipotent** means upper triangular with 1’s on the diagonal. Any unipotent matrix can be written in the form $I_n + X$ where $I_n$ is the $n \times n$ identity matrix and $X$ is a strictly upper triangular matrix, i.e.,
\[
x_{ij} = 0 \text{ unless } j \geq i + 1
\]
Let $U_k$ be the set of strictly upper triangular matrices $X = (x_{ij})$ so that
\[
x_{ij} = 0 \text{ unless } j \geq i + k
\]
Then
\[
U_j U_k \subseteq U_{j+k}
\]
Therefore, every element of $U_k, k \geq 1$ is nilpotent: $X^n = 0$. This means that
\[
(I + X)^{-1} = I - X + X^2 - X^3 + \cdots + (-1)^{n-1}X^{n-1} = I - X(I + X)^{-1}.
\]
Let
\[
T_k = \{I + X \mid X \in U_k\}.
\]
Then $T(n,\mathbb{Z}) = \{I + X \mid X \in U_1\} = T_1$.

**Lemma 2.5.** $T_1 \supseteq T_2 \supseteq T_3 \supseteq \cdots \supseteq T_n = \{I\}$ is a normal tower with abelian subquotients.
Proof. Take arbitrary elements $I + X, I + Y$ in $T_1, T_k$ resp. Then
$$(I + X)(I + Y)(I + X)^{-1} = (I + X + Y + XY)(I + X)^{-1} = I + (Y + XY)(I + X)^{-1}$$
This is an element of $T_k$ since $(Y + XY)(I + X)^{-1} \in U_k U_0 \subseteq U_k$.
Therefore, $T_k \triangleleft T_1$.

If $X, Y \in U_k$ then expanding $(I + X)^{-1}$ as $I - X(I + X)^{-1}$ we get:
$$(I + X)(I + Y)(I + X)^{-1} = I + Y + XY - (Y + XY)X(I + X)^{-1}.$$ 
$[I + X, I + Y] = I + (XY - (Y + XY)X(I + X)^{-1})(I + Y)^{-1} \in T_{2k}$
Therefore, $T_k/T_{2k}$ is abelian. \hfill \Box

This shows that $T(n, \mathbb{Z})$ is solvable of degree $\leq k$ if $n \leq 2^k$.

2.3. subgroups and quotient groups. We want to know that subgroups and quotient groups of solvable groups are solvable.

**Theorem 2.6.** Every subgroup of a solvable group is solvable.

Proof. If $H \leq G$ then $H' \leq G'$. This, in turn, implies that $(H')' = H^{(2)} \leq G^{(2)} = (G')'$. Eventually we get $H^{(n)} \leq G^{(n)} = \{e\}$. So, $H$ is soluble of degree $\leq n$. \hfill \Box

**Theorem 2.7.** Every quotient group of a solvable group is solvable.

Proof. I claim that
$$G / N \triangleright G' N / N \triangleright G^{(2)} N / N \triangleright \cdots \triangleright G^{(n)} N / N = \{e\}$$
is a normal tower for $G / N$ with abelian quotients. In fact $G^{(k)} N / N$ is the image of $G^{(k)}$ under the homomorphism
$$G^{(k)} \hookrightarrow G \twoheadrightarrow G / N.$$ 
Therefore,
$$G^{(k)} N / N = (G / N)^{(k)}.$$ 
This uses the following lemma. \hfill \Box

**Lemma 2.8.** If $\phi : G \twoheadrightarrow H$ is an epimorphism (surjective homomorphism) then $\phi(G^{(k)}) = H^{(k)}$ for all $k \geq 0$.

**Theorem 2.9.** Suppose that $N \triangleleft G$ and $N, G / N$ are solvable. Then $G$ is solvable.

Proof. We are given that $N, G / N$ are solvable. So, we have abelian towers
$$N \triangleright N_1 \triangleright N_2 \cdots N_n = \{e\}$$
Since subgroups of $G / N$ all have the form $H / N$ for some $N \leq H \leq G$, the abelian tower for $G / N$ looks like this:
$$G / N \triangleright G_1 / N \triangleright G_2 / N \cdots G_m / N = \{e\}$$
Where $G_m = N$. By the following lemma the abelian subquotients of this tower are
\[
\frac{G^{(k)}/N}{G^{(k+1)}/N} \cong \frac{G^{(k)}}{G^{(k+1)}}
\]
Therefore,
\[
G \supset G_1 \supset G_2 \supset \cdots \supset G_m = N \supset N_1 \supset \cdots \supset N_n = \{e\}
\]
is a normal tower for $G$ with abelian subquotients proving that $G$ is solvable of degree $\leq n + m$. □

**Lemma 2.10.** If $N, H$ are normal subgroups of $G$ with $N \leq H$ then
\[
\frac{G/N}{H/N} \cong \frac{G}{H}.
\]
**Proof.** Let $\phi : G/N \to G/H$ be the homomorphism given by $\phi(aN) = aNH = aH$. Then $\phi$ is clearly onto and $\ker \phi = \{aN \in G/N \mid aH = H\}$. But $aH = H$ iff $a \in H$. So, $\ker \phi = H/N$. The lemma follows from the equation
\[
\text{image}(\phi) = \frac{\text{domain}(\phi)}{\ker \phi}
\]
□