13 Schur-Zassenhaus Lemma

The Schur-Zassenhaus Lemma says that when $Q$ and $K$ are finite of relatively prime order then any extension of $K$ by $Q$ is a semidirect product. We have already proved this in the case when $Q$ and $K$ are solvable. [Why is that?] In this section we prove it without the solvability assumption using factor sets.

Theorem 13.1. Every abelian normal Hall subgroup $K$ of a finite group $G$ has a complement.

Remark 13.2. Using what we learned in the previous section we can rephrase this relatively simple statement in the following terms (obfuscating it meaning but facilitating its proof):

If $Q$ is a finite group of order $n$ and $K$ is a finite $Q$-module of order $m$ where $(n, m) = 1$ then $H^2(Q, K) = 0$, i.e., every extension of $K$ by $Q$ is a semidirect product.

Proof. The key to this proof lies in the fact that multiplication by $n = |Q|$ is an automorphism of $K$. [Since $(n, m) = 1$, there are integers $s, t$ so that $sm + tn = 1$. Since multiplication by $m = |K|$ is zero on $K$ this means that $tn = id_K$.]

By Remark 13.2 we need to show that every factor set $f : Q \times Q \to K$ is a coboundary. This easy. Let $h : Q \to K$ be given by

$$h(x) = t \sum_{y \in Q} f(x, y)$$

This is defined since $Q$ is finite and $t \in \mathbb{Z}$. Also, if $x = 1$ then each $f(1, y) = 0$ so $h(1) = 0$. Since $f$ is a cocycle, it satisfies:

$$xf(y, z) - f(xy, z) + f(x, yz) = f(x, y).$$

Summing over all $z \in Q$ we get:

$$x \sum_{z \in Q} f(y, z) - \sum_{z \in Q} f(xy, z) + \sum_{w \in Q} f(x, w) = nf(x, y).$$

In the third sum $w = yz$ runs over all the elements of $Q$ as $z$ runs over the elements of $Q$. Multiply each term by $t$ and we get:

$$xh(y) - h(xy) + h(x) = tnf(x, y) = f(x, y),$$

i.e., $f = \delta h$. □

Since the proof did not use the fact that $K$ is finite we get:

Corollary 13.3. If $Q$ is finite and multiplication by $n = |Q|$ is an automorphism of $K$ then $H^2(Q, K) = 0$. 

For example, $H^2(Q,K) = 0$ if $Q$ is finite and $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

**Theorem 13.4.** If $K$ is an abelian normal Hall subgroup of a finite group $G$ then any two complements of $K$ are conjugate.

As in the case of Theorem 13.1, this theorem is equivalent to the equation $H^1(Q,K) = 0$ by the following lemma and definitions.

**Definition 13.5.** A 1-cocycle for $Q$ with coefficients in $K$ is defined to be a mapping $h : Q \to K$ so that $\delta h = 0$. This implies $h(1) = \delta h(1,1) = 0$ so $h$ is automatically normalized. The group of all such $h$ is denoted $Z^1(Q,K)$.

If $a \in K$ then let $\delta a : Q \to K$ be given by

$$\delta a(x) = xa - a.$$ 

This is a 1-cocycle which we call a 1-coboundary. [Also known as principal crossed homomorphism.]

Let $B^1(Q,K) \leq Z^1(Q,K)$ be the group of 1-coboundaries. Then the 1-dimensional cohomology of $Q$ with coefficients in $K$ is defined to be the quotient:

$$H^1(Q,K) = \frac{Z^1(Q,K)}{B^1(Q,K)}$$

**Lemma 13.6.** There is a bijection between $H^1(Q,K)$ and the set of conjugacy classes of subgroups $H \leq K \rtimes Q$ which are complementary to $Q$.

Suppose for a moment that this is true.

**Proof of Theorem 13.4.** By the lemma it is enough to show that $H^1(Q,K) = 0$, i.e., that every 1-cocycle $h : Q \to K$ is of the form $\delta a$ for some $a \in K$. This is the same proof as in Theorem 13.1. Let

$$a_0 = t \sum_{x \in Q} h(x).$$

The cocycle condition $\delta h = 0$ gives:

$$xh(y) - h(xy) = h(x)$$

Summing over all $y \in Q$ and multiplying by $t$ we get:

$$xa_0 - a_0 = tnh(x) = h(x),$$

i.e., $h = \delta a_0$.

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1This is also called a crossed homomorphism since the condition $\delta h$ says that

$$h(xy) = h(x) + xh(y).$$

In fact, if the action is trivial, $h$ is a homomorphism iff $\delta h = 0$.
Proof of Lemma 13.6. Let $G = K \rtimes Q = \{(a, x)\}$. If $H \leq G$ is complementary to $K$ then $H$ is the image of a section $s : Q \to G$ of $\pi : G \to Q$. Thus

$$H = \{s(x) = (h(x), x) \mid x \in Q\}.$$ 

Since $s$ is a homomorphism, $h$ is a cocycle:

$$s(x)s(y) = (h(x), x)(h(y), y) = (h(x) + xh(y), xy)$$
$$s(xy) = (h(xy), xy)$$

These are equal iff $\delta h = 0$. Thus $H \leftrightarrow h$ gives a bijection:

$$Z^1(Q, K) \cong \{H \leq K \rtimes Q \mid H \text{ is complementary to } K\}.$$ 

If $a \in K \leq K \rtimes Q$ and $H = \{(h(x), x)\}$ then $aHa^{-1}$ is the set of all

$$(a, 1)(h(x), x)(-a, 1) = (a + h(x) - xa, x).$$

Thus the 1-cocycle corresponding to $aHa^{-1}$ is $h - \delta a$. So the cosets of $B^1(Q, K)$ in $Z^1(Q, K)$ correspond to the $K$-conjugacy classes of complements $H$ to $K$. But these are the same as the $G$ conjugacy classes since $G = KH$.

Theorem 13.7 (Schur-Zassenhaus Lemma). Every normal Hall subgroup $K$ of a finite group $G$ has a complement.

Proof. Note that any subgroup of $G$ of order $n = |Q|$ is a complement to $K$. The proof is by induction on $m = |K|$. If $m = 1$ then $H = G$ is the complement. So suppose $m > 1$.

Suppose first that there exists $1 < T < K$ so that $T \triangleleft G$. Then $K/T$ is a normal Hall subgroup of $G/T$ with $|K/T| < m$. Thus $K/T$ has a complement $H/T \cong (G/T)/(K/T) \cong G/K = Q$. Since $T$ is a normal Hall subgroup of $H$ it has a complement $C \cong H/T \cong Q$. Since $C \leq H \leq G$, $C$ is a complement of $K$ in $G$. Thus we may assume that $K$ is a minimal normal subgroup of $G$.

Let $P$ be a Sylow subgroup of $K$. Then the Frattini argument tells us that $G = KN_G(P)$. Since $K \cap N_G(P) = N_K(P)$ we get the following picture.

$$\begin{tikzpicture}
  \node (G) at (0,0) {$G$};
  \node (K) at (0,-1) {$K$};
  \node (NGP) at (1,-1) {$N_G(P)$};
  \node (NKp) at (1,-2) {$N_K(P)$};
  \draw (K) to (NGP);
  \draw (K) to (NKp);
\end{tikzpicture}$$

By the second isomorphism theorem we have

$$G/K \cong N_G(P)/N_K(P) \cong Q.$$ 

So, $N_K(P)$ is a normal Hall subgroup of $N_G(P)$. If $|N_K(P)| < m$ then $N_K(P)$ has a complement $H \cong N_G(P)/N_K(P) \cong Q$ so $H$ is a complement of $K$ in $G$. 

Thus we may assume that $N_K(P) = K$ and $N_G(P) = G$, i.e., $P \trianglelefteq G$. Since $K$ is a minimal normal subgroup we must have $K = P$. If $K$ is nonabelian then $1 < Z(P) < P = K$ and $Z(P) \text{ char } P \trianglelefteq G$ implies $Z(P) \vartriangleleft G$ contradicting the minimality of $K$. Thus $K$ is abelian and the theorem follows from 13.1.

HW.05.ex04: If $Q$ acts trivially on $K$ then show that $H^1(Q, K) \cong \text{Hom}(Q, K)$.

HW.05.ex05: Suppose that $Q$ is a finite group and $K$ is a finite $Q$-module so that $n = |Q|$ and $m = |K|$ are relatively prime. Suppose that $H$ is another finite group so that $|H|$ is relatively prime to $n$ (but not to $m$). Suppose further that $H^2(H, K) = 0$ where $H$ acts trivially on $K$. Then show that $H^2(H \times Q, K) = 0$. 
