Injective envelope

There is one other very important fact about injective modules which was not covered in class for lack of time and which is also not covered in the book. This is the fact that every $R$-module $M$ embeds in a minimal injective module which is called the injective envelope of $M$. This is from Jacobson’s Basic Algebra II.

**Definition 5.5.** An embedding $A \hookrightarrow B$ is called essential if every nonzero submodule of $B$ meets $A$. I.e., $C \subseteq B, C \neq 0 \Rightarrow A \cap C \neq 0$.

For example, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is essential because, if a subgroup of $\mathbb{Q}$ contains $a/b$, then it contains $a \in \mathbb{Z}$. Also, every isomorphism is essential.

**Exercise 5.6.** Show that the composition of essential maps is essential.

**Lemma 5.7.** Suppose $A \subseteq B$. Then

1. $\exists X \subseteq B$ s.t. $A \cap X = 0$ and $A \hookrightarrow B/X$ is essential.
2. $\exists C \subseteq B$ maximal so that $A \subseteq C$ is essential.

**Proof.** For (1) the set of all $X \subseteq B$ s.t. $A \cap X = 0$ has a maximal element by Zorn’s lemma. Then $A \hookrightarrow B/X$ must be essential, otherwise there would be a disjoint submodule of the form $Y/X$ and $X \subset Y, A \cap Y = 0$ contradicting the maximality of $Y$. For (2), $C$ exists by Zorn’s lemma. \qed

**Lemma 5.8.** $\mathbb{Q}$ is injective iff every short exact sequence

$$0 \to \mathbb{Q} \to M \to N \to 0$$

splits.

**Proof.** If $\mathbb{Q}$ is injective then the identity map $\mathbb{Q} \to \mathbb{Q}$ extends to a retraction $r : M \to \mathbb{Q}$ giving a splitting of the sequence. Conversely, suppose that every sequence as above splits. Then for any monomorphism $i : A \hookrightarrow B$ and any morphism $f : A \to \mathbb{Q}$ we can form the pushout $M$ in the following diagram

$$A \xrightarrow{i} B \xleftarrow{f} Q \xrightarrow{j} M \xrightarrow{f'} M$$

As you worked out in your homework, these morphisms form an exact sequence:

$$A \xrightarrow{(f)} \mathbb{Q} \oplus B \xrightarrow{(j,f')} M \to 0$$
Since \( i \) is a monomorphism by assumption, \( A \) is the kernel of \((j, -f')\). Therefore (again using your homework) \( A \) is the pull-back in the above diagram. This implies that \( j \) is a monomorphism. [Any morphism \( g : X \to Q \) which goes to zero in \( M \), i.e., so that \( j \circ g = 0 \), will give a morphism \((g, 0) : X \to Q \oplus B \) which goes to zero in \( M \) and therefore lifts uniquely to \( h : X \to A \) so that
\[
\begin{pmatrix} f \\ i \end{pmatrix} \circ h = \begin{pmatrix} f \circ h \\ i \circ h \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}
\]
But \( i \) is a monomorphism. So, \( i \circ h = 0 \) implies \( h = 0 \) which in turn implies that \( f \circ h = g = 0 \). So, \( j \) is a monomorphism.]

Since \( j \) is a monomorphism there is a short exact sequence
\[
0 \to Q \xrightarrow{j} M \to \text{coker } j \to 0
\]
We are assuming that all such sequences split. So, there is a retraction \( r : M \to Q \). \((r \circ j = \text{id}_Q)\) Then it is easy to see that \( r \circ f' : B \to Q \) is the desired extension of \( f : A \to Q \):
\[
r \circ f' \circ i = r \circ j \circ f = \text{id}_Q \circ f = f
\]
So, \( Q \) is injective. \( \square \)

**Lemma 5.9.** \( Q \) is injective if and only if every essential embedding \( Q \hookrightarrow M \) is an isomorphism.

**Proof.** \((\Rightarrow)\) Suppose \( Q \) is injective and \( Q \hookrightarrow M \) is essential. Then the identity map \( Q \to Q \) extends to a retraction \( r : M \to Q \) whose kernel is disjoint from \( Q \) and therefore must be zero making \( M \cong Q \).

\((\Leftarrow)\) Now suppose that every essential embedding of \( Q \) is an isomorphism. We want to show that \( Q \) is injective. By the previous lemma it suffices to show that every short exact sequence
\[
0 \to Q \xrightarrow{j} M \to N \to 0
\]
splits. By Lemma 5.7 there is a submodule \( X \subseteq M \) so that \( jQ \cap X = 0 \) and \( Q \hookrightarrow M/X \) is essential. Then, by assumption, this map must be an isomorphism. So, \( M \cong Q \oplus X \) and the sequence splits proving that \( Q \) is injective. \( \square \)

**Theorem 5.10.** For any \( R \)-module \( M \) there exists an essential embedding \( M \hookrightarrow Q \) with \( Q \) injective. Furthermore, \( Q \) is unique up to isomorphism under \( M \).

**Proof.** We know that there is an embedding \( M \hookrightarrow Q_0 \) where \( Q_0 \) is injective. By Lemma 5.7 we can find \( Q \) maximal with \( M \hookrightarrow Q \hookrightarrow Q_0 \) so that \( M \hookrightarrow Q \) is essential.
Claim: $Q$ is injective.

If not, there exists an essential $Q \hookrightarrow N$. Since $Q_0$ is injective, there exists $f : N \rightarrow Q_0$ extending the embedding $Q \hookrightarrow Q_0$. Since $f$ is an embedding on $Q$, $\ker f \cap Q = 0$. This forces $\ker f = 0$ since $Q \hookrightarrow N$ is essential. So, $f : N \rightarrow Q_0$ is a monomorphism. This contradicts the maximality of $Q$ since the image of $N$ is an essential extension of $M$ in $Q_0$ which is larger than $Q$.

It remains to show that $Q$ is unique up to isomorphism. So, suppose $M \hookrightarrow Q'$ is another essential embedding of $M$ into an injective $Q'$. Then the inclusion $M \hookrightarrow Q'$ extends to a map $g : Q \rightarrow Q'$ which must be a monomorphism since its kernel is disjoint from $M$. Also, $g$ must be onto since $g(Q)$ is injective making the inclusion $g(Q) \hookrightarrow Q'$ split which contradicting the assumption that $M \hookrightarrow Q'$ is essential unless $g(Q) = Q'$. \qed