5. Integral dependence and valuation

This section is about integral extensions and the integral closure of a ring.

5.1. Integral extensions.

Definition 5.1. Suppose that $A$ is a subring of $B$. We say that $B$ is an integral extension of $A$ if very element of $B$ is integral over $A$. Recall that $b \in B$ is integral over $A$ if $b$ is a root of a monic polynomial

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

with coefficients $a_i \in A$. (In other words, $f(b) = 0$.)

If $f(b) = 0$ then every power of $b$ can be written as an $A$-linear combination of the powers

$$1, b, b^2, \ldots, b^{n-1}$$

So, every element of $A[b]$ can be written as an $A$-linear combination of these $n$ elements.

Lemma 5.2. $A[b]$ is finitely generated as an $A$-module (if $b$ is integral over $A$).

The set of all elements of $B$ which are integral over $A$ is called the integral closure of $A$ in $B$ and we call it $C$. We will show that $C$ is a subring of $B$ containing $A$. We say that $A$ is integrally closed in $B$ if $C = A$.

Example 5.3. $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

To prove this take any element of $\mathbb{Q}$ which is integral over $\mathbb{Z}$ and write it in reduced form: $x/s$ where $x, s$ are relatively prime. Then

$$\left(\frac{x}{s}\right)^n + a_1 \left(\frac{x}{s}\right)^{n-1} + \cdots + a_n = 0$$

$$x^n + a_1x^{n-1}s + \cdots + a_ns^n = 0$$

This implies that $s$ divides $x^n$. So, $s = \pm 1$.

Exercise 5.4. Show that any UFD is integrally closed in its field of fractions.

Here are more things that follow just from the definition:

Proposition 5.5. Suppose that $B$ is an integral extension of $A$.

1. Let $a = A \cap b$ (so that $A/a \subseteq B/b$) then $B/b$ is an integral extension of $A/a$.

2. $S^{-1}B$ is an integral extension of $S^{-1}A$. 
Proposition 5.6. Suppose that $A$ is a subring of $B$ and $b \in B$. Then the following are equivalent.

1. $b$ is integral over $A$.
3. $A[b]$ is contained in a subring $M$ of $B$ which is a f.g $A$-module.
4. There is a f.g. $A$-module $M$ on which $A[b]$ acts faithfully.

$A[b]$ acts faithfully on $M$ means that the induced ring homomorphism $A[b] \to \text{End}(M)$ is injective. This is equivalent to saying that the annihilator of $M$ in $A[b]$ is zero.

Proof. (1) $\Rightarrow$ (2) by the lemma and (2) $\Rightarrow$ (3) is clear. Also (3) $\Rightarrow$ (4) since $A[b]$ acts faithfully on $M$ because $M$ contains $A[b]$: The composition $A[b] \to \text{End}(M) \to \text{End}(A[b])$ is injective. So, it suffices to prove that (4) $\Rightarrow$ (1). This follows from the determinant trick (Prop 2.3). Multiplication by $b$ induces an $A$-module homomorphism $\varphi : M \to aM = M$ where $a = A$. Therefore there is a monic polynomial $f[x] \in A[x]$ so that $f(\varphi) = 0$ in $\text{End}(M)$. Since $M$ is a faithful $A[b]$ module, this implies that $f(b) = 0$ in $A[b]$. So, $b$ is integral over $A$.

Corollary 5.7. If $A$ is a subring of $B$ and $B$ is a finite $A$-algebra (f.g. $A$-module) then $B$ is an integral extension of $A$.

Proof. Use (4).

Corollary 5.8. If $b_1, \cdots, b_n \in B$ are integral over $A$ then $A[b_1, \cdots, b_n]$ is an integral extension of $A$.

Proof. Suppose that $f_i(x) \in A[x]$ are monic polynomials of degree $d_i$ so that $f_i(b_i) = 0$. Then any power of $b_i$ can be written as an $A$-linear combination of the powers $1, b_i, b_i^2, \cdots, b_i^{d_i-1}$.

So, every element of $A[b_1, \cdots, b_n]$ can be written as an $A$-linear combination of the monomials $b_1^{m_1}b_2^{m_2}\cdots b_n^{m_n}$ where $m_i < d_i$. So, $A[b_1, \cdots, b_n]$ is f.g. $A$-module. So, all of its elements are integral over $A$.

Corollary 5.9. If $B$ is an integral extension of $A$ and $C$ is an integral extension of $B$ then $C$ is an integral extension of $A$.

Corollary 5.10. If $A \subseteq C \subseteq B$ and $C$ is the integral closure of $A$ in $B$ then $C$ is integrally closed.