0.3. **systems of first order equations.** I explained that differential equations involving second and higher order derivatives can be reduced to a system of first order equations by introducing more variables. Then I did the following example.

\[ y' = z \]
\[ z' = 6y - z \]

In matrix form this is:

\[
\frac{d}{dt} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}
\]

Which we can write as \( Y = AY \) with

\[
Y = \begin{pmatrix} y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}
\]

0.3.1. **exponential of a matrix.** The solution of this equations is

\[ Y = e^{tA}Y_0 \]

where \( Y_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \) and

\[ e^{tA} := I_2 + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \cdots \]

This works because the derivative of each term is \( A \) times the previous term:

\[
\frac{d}{dt} \frac{k!}{k!} t^k A^k = \frac{k t^{k-1} A^k}{k!} = \frac{t^{k-1} A^k}{(k-1)!} = A \frac{t^{k-1} A^{k-1}}{(k-1)!}
\]

So,

\[
\frac{d}{dt} e^{tA} = A e^{tA}
\]

0.3.2. **diagonalization (corrected).** Then I explained how to compute \( r^{tA} \). You have to diagonalize \( A \). This means

\[ A = QDQ^{-1} \]

where \( D \) is a diagonal matrix \( D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \).

I should have explained this formula so that I get it right: If \( X_1, X_2 \) are eigenvalues of \( A \) with eigenvalues \( d_1, d_2 \) then \( AX_1 = X_1d_1, AX_2 = X_2d_2 \) and

\[ A(X_1X_2) = (X_1d_1X_2d_2) = (X_1X_2) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \]
Solve for $A$ gives

$$A = (X_1 X_2) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} (X_1 X_2)^{-1} = Q D Q^{-1}$$

where $Q = (X_1, X_2)$.

This is a good idea because $A^2 = Q D Q^{-1} Q D Q^{-1} = Q D^2 Q^{-1}$ and more generally, $t^k A^k = t^k Q D^k Q^{-1}$. Divide by $k!$ and sum over $k$ to get:

$$e^{tA} = Q e^{tD} Q^{-1} = Q \begin{pmatrix} e^{td_1} & 0 \\ 0 & e^{td_2} \end{pmatrix} Q^{-1}$$

0.3.3. *eigenvectors and eigenvalues.* The diagonal entries $d_1, d_2$ are the eigenvalues of the matrix $A$ and $Q = (X_1, X_2)$ where $X_i$ is the eigenvector corresponding to $d_i$. This works if the eigenvalues of $A$ are distinct. The eigenvalues are defined to be the solutions of the equation

$$\det(A - \lambda I) = 0$$

but there is a trick to use for $2 \times 2$ matrices. The determinant of a matrix is always the product of its eigenvalues:

$$\det A = d_1 d_2 = -6$$

The trace (sum of diagonal entries) is equal to the sum of the eigenvalues:

$$trA = d_1 + d_2 = -1$$

So, $d_1 = 2, d_2 = -3$. The eigenvalues are $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

So

$$Q = (X_1 X_2) = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\det Q} \begin{pmatrix} -3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix}$$

The solution to the original equation is

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$
0.4. **Linear difference equations.** We are looking for a sequence of numbers \( f(n) \) where \( n \) ranges over all the integers from \( K \) to \( N \) (\( K \leq n \leq N \)) so that

\[
0.3 \quad f(n) = af(n - 1) + bf(n + 1)
\]

I pointed out that the solution set is a vector space of dimension 2. So we just have to find two linearly independent solutions. Then I followed the book.

The solution has the form \( f(n) = c^n \) where you have to solve for \( c \):

\[
0.4 \quad c^n = ac^{n-1} + bc^{n+1}
\]

\[
bc^2 - c + a = 0
\]

\[
c = \frac{1 \pm \sqrt{1 - 4ab}}{2b}
\]

There were two cases.

**Case 1:** \( (4ab \neq 1) \) When the quadratic equation (0.4) has two roots \( C_1, C_2 \) then the linear combinations of \( c_1^n \) and \( c_2^n \) give all the solutions of the homogeneous linear recursion (0.3).

**Case 2:** \( (4ab = 1) \) In this case there is only one root \( c = \frac{1}{2b} \) and the two independent solutions are \( f(n) = c^n \) and \( nc^n \). The reason we get a factor of \( n \) is because when a linear equation has a double root then this root will also be a root of the derivative. This gives \( f(n) = nc^{n-1} \) as a solution. But then you can multiply by the constant \( c \) since the equation is homogeneous.

**Example 0.4. (Fibonacci numbers)** These are given by \( f(0) = 1, f(1) = 1 \) and \( f(n + 1) = f(n) + f(n - 1) \) or:

\[
f(n) = f(n + 1) - f(n - 1)
\]

This is \( a = -1, b = 1 \). The roots of the quadratic equation are \( c = \frac{1 \pm \sqrt{5}}{2} \).

So,

\[
f(n) = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]

This is a rational number since it is Galois invariant (does not change if you switch the sign of \( \sqrt{5} \)). However, it is not clear from the formula why it is an integer.