1.6.2. *expected time.* Today I did a more thorough explanation of the expected time until we reach a recurrent state. I started by reviewing the basics of substochastic matrices.

**Definition 1.20.** A *substochastic matrix* is a square matrix $Q$ with nonnegative entries so that every row adds up to at most 1.

For example,

$$Q = \begin{pmatrix} 1/2 & 1/3 \\ 1/4 & 3/4 \end{pmatrix}$$

is substochastic.

Given any subset $C$ of the set of states $S$, the $C$-to-$C$ transition matrix will always be substochastic.

**Lemma 1.21.** If $C$ contains no recurrent class then

- (a) $I + Q + Q^2 + Q^3 + \cdots$ converges to $(I - Q)^{-1}$.
- (b) $I + 2Q + 3Q^2 + 4Q^3 + \cdots$ converges to $(I - Q)^{-2}$.

**Proof.** (a) follows from the computation

$$(I - Q)(I + Q + Q^2 + Q^3 + \cdots) = I.$$  

For (b), the argument is:

$$(I - Q)(I + 2Q + 3Q^2 + 4Q^3 + \cdots) = I + 2Q + 3Q^2 + 4Q^3 + \cdots - Q - 2Q^2 - 3Q^3 - \cdots = I + Q + Q^2 + Q^3 \cdots = (I - Q)^{-1}$$

by (a). Therefore,

$$I + 2Q + 3Q^2 + 4Q^3 + \cdots = \frac{(I - Q)^{-1}}{I - Q} = (I - Q)^{-2}.$$

I gave another example to illustrate both of these formulas.
In this example, there is only one recurrent class \( R_1 = \{2, 3, 4\} \). The set \( C = \{1, 2\} \) contains the recurrent state 2 but it does not contain a recurrent class \((R_1 \text{ is not contained in } C)\). Therefore, the lemma applies.

Inserting the implied loop at vertex 2, we saw that the substochastic matrix is

\[
Q = \begin{pmatrix}
0 & 3/4 \\
0 & 1/2
\end{pmatrix}
\]

Then

\[
I - Q = \begin{pmatrix}
1 & -3/4 \\
0 & 1/2
\end{pmatrix}
\]

So,

\[
(I - Q)^{-1} = \begin{pmatrix}
1 & 3/2 \\
0 & 2
\end{pmatrix}, \quad (I - Q)^{-2} = \begin{pmatrix}
1 & 9/4 \\
0 & 4
\end{pmatrix}.
\]

These numbers are used to answer questions such as the following.

Question 1: If you start at 1, what is the probability that you reach 4 before you reach 3?

The first step in answering this question is to realize that: It does not matter what happens after you reach 3 or 4 because, at that point, the question has been answered. So, the numbers

\[
p(3, 2) = 1/3, p(4, 3) = 1/2
\]

are irrelevant and we can replace them with 0. In other words, we can make 3 and 4 into absorbing states. This simplification process gives a new probability transition matrix which we put into canonical form:

\[
\tilde{P} = \begin{pmatrix}
3 & 4 & 1 & 2 \\
1 & 0 & 0 & 0 \\
1 & 1/4 & 0 & 3/4 \\
1/4 & 1/4 & 0 & 1/2
\end{pmatrix}
\]

\[
S_1 = \begin{pmatrix}
1/4 \\
1/2
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
1/2
\end{pmatrix}
\]

\[
S_T = S_1 + S_2 = \begin{pmatrix}
1/4 \\
1/2
\end{pmatrix}.
\]

Answer: The answer to the question is given by the first coordinate of the vector

\[
(I - Q)^{-1}S_1 = \begin{pmatrix}
1 & 3/2 \\
0 & 2
\end{pmatrix} \begin{pmatrix}
1/4 \\
1/2
\end{pmatrix} = \begin{pmatrix}
1 + 3/4 \\
2/4
\end{pmatrix} = \begin{pmatrix}
5/8 \\
1/2
\end{pmatrix}
\]
which is $5/8$. (If we started at state 2, the answer would be the second coordinate which is $1/2$.) This was proved on Monday (Theorem 1.18).

The matrix $(I - Q)^{-2}$ is used to answer a different question.

**Question 2:** If we start at $X_0 = 1$, how long will it take to reach 3 or 4? In other words, we want to calculate the conditional expected value

$$E(T \mid X_0 = 1) = ?$$

of

$$T := \text{smallest } n \text{ so that } X_n = 3 \text{ or } 4.$$  

**Answer:** Just take the definition of expected value:

$$E(T \mid X_0 = 1) := \sum_{n=0}^{\infty} n \mathbb{P}(T = n \mid X_0 = 1).$$

But

$$\mathbb{P}(T = n \mid X_0 = 1) = (Q^{n-1} S_T)_1$$

is the first coordinate of

$$Q^{n-1} S_T$$

because $T = n$ means we stay in the set $C$ for $n - 1$ turns and then go to one of the recurrent states 3, 4 (according to the modified transition matrix $\tilde{P}$). $Q^{n-1}$ gives the probability of stay in $C$ for $n - 1$ turns and $S_T$ gives the probability of moving on the $n$th turn from $C$ to 3 or 4.

$$E(T \mid X_0 = 1) := \sum_{n=0}^{\infty} n \mathbb{P}(T = n \mid X_0 = 1)$$

$$= \sum_{n=0}^{\infty} n(Q^{n-1} S_T)_1 = ((I - Q)^{-2} S_T)_1$$

$$= \begin{pmatrix} 1 & 9/2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 2 \end{pmatrix}$$

So, the answer is

$$E(T \mid X_0 = 1) = 5/2.$$