1.7. **Transient and recurrent.** In a finite Markov chain, every state is either transient or recurrent depending on whether it is in a transient or recurrent communication class. Here is a summary of the differences.

1.7.1. *transient classes.*

1. If \( i \) is a transient class then you will a.s. visit \( i \) only finitely many times.

2. If \( i, j \) are transient states then the expected number of visits to \( j \), starting at \( i \)

\[
\mathbb{E}(\text{visits to } j \mid X_0 = i) = \sum_{n=0}^{\infty} \mathbb{P}(X_n = j \mid X_0 = i)
\]

is equal to the \((i, j)\) coordinate of the matrix

\[
I + Q + Q^2 + Q^3 + \cdots = (I - Q)^{-1}.
\]

3. Starting at a transient state \( i \), the expected value of \( T = \) the number of steps to reach a recurrent state is the \( i \)th coordinate of

\[
(I - Q)^{-2}S_T
\]

4. The expected length of time to return to \( i \) given that \( X_0 = i \) is

\[
\mathbb{E}(\text{smallest } n > 0 \text{ so that } X_n = i \mid X_0 = i) = \infty.
\]

The reason is that there is a nonzero chance \( p \) that you could be waiting forever. Then \( \mathbb{E} = (\cdots) + p \cdot \infty = \infty \)

1.7.2. *recurrent classes.*

1. If \( j \) is recurrent then you will a.s. return to \( j \) an infinite number of times if \( X_0 = j \).

2. Every recurrent class \( R_i \) has an invariant distribution \( \pi_i \).

3. The long term probability of being in state \( j \in R_i \) given that \( X_0 \in R_i \) is equal to the \( j \)-coordinate \( \pi_i(j) \) of \( \pi_i \).

4. The expected number of visits to \( j \) is infinite:

\[
\mathbb{E}(\text{number of visits to } j \mid X_0 = j) = \infty.
\]

5. The expected length of time between visits to \( j \) is

\[
\mathbb{E}(\text{smallest } n > 0 \text{ so that } X_n = j \mid X_0 = j) = \frac{1}{\pi_i(j)}.
\]

(For example if \( \pi_i(j) = 1/3 \) then you spend 1/3 of the time at \( j \) and the average time between visits is 3. Since this is obvious I won’t prove it.)