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2. Countable Markov Chains

We will now talk about *countably infinite* Markov chains. The concept is the same but having an infinite number of states makes several qualitative differences.

2.1. Concept and examples. We will study Markov chains with countably infinite number of states. For example, we could take the nonnegative integers:

\[ S = \{0, 1, 2, 3, 4, \ldots\} \]

or all integers:

\[ S = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \]

We will also study the set of integer points in \( n \)-dimensional space:

\[ S = \mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \text{each } x_i \in \mathbb{Z}\} \]

This is called the *integer lattice* in \( \mathbb{R}^n \).

The *transition matrix* \( P \) is now infinite:

\[ P = (p_{ij}) = (p(i, j)) \]

The numbers still mean the same thing:

\[ p_{ij} = p(i, j) = \Pr(X_{n+1} = j \mid X_n = i) \]

And they still satisfy the condition that \( 0 \leq p_{ij} \leq 1 \) and the rows add up to one. But now this means that we have an infinite series adding to 1. For example, if \( S = \{0, 1, 2 \ldots\} \) then the condition is:

\[ p_0 + p_1 + p_2 + p_3 + \cdots = 1 \]

In the general case, the condition is:

\[ \sum_j p_{ij} = 1. \]

2.1.1. example: random walk on \( \mathbb{Z} \). In the first example I only posed the question but didn’t answer it. This is the example where \( S = \mathbb{Z} \) and, at each point, there is a probability of \( p \) of going to the right and \( 1 - p \) of going to the left:

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ 1-p \quad p \]

\[ \Pr(X_{n+1} = x + 1 \mid X_n = x) = p(x, x + 1) = p \]

\[ \Pr(X_{n+1} = x - 1 \mid X_n = x) = p(x, x - 1) = 1 - p \]
The question is: If \( p \geq \frac{1}{2} \) then how certain is it that you will go to \(+\infty\)? You might want to write the question as:

\[ \mathbb{P}(\lim_{n \to \infty} X_n = \infty) = ? \]

However, this doesn’t quite make sense because the limit may not exist. The Markov chain might (and will) oscillate wildly. The correct question is:

\[ \mathbb{P}(\lim \sup X_n = \infty) = ? \]

**Definition 2.1.** For any sequence or real numbers \( x_n \),

\[ \lim \sup x_n := \lim_{n \to \infty} (\sup \{x_n, x_{n+1}, x_{n+2}, \ldots\}) \]

where “\( \sup \)” is short for supremum which means maximum for a finite set and least upper bound for an infinite set.

For any sequence of real numbers \( x_0, x_1, \cdots \) the suprema

\[ \sup \{x_0, x_1, \cdots \} \geq \sup \{x_1, x_2, \cdots \} \geq \sup \{x_2, x_3, \cdots \} \geq \cdots \]

form a nonincreasing sequence of real numbers which may all be \(+\infty\). Therefore, they always have a limit in the set of real numbers union with \(+\infty\):

\[ \lim \sup x_n \in [-\infty, +\infty]. \]

(Any bounded nonincreasing sequence has a limit.)

Similarly, the “\( \liminf \)” is defined by

\[ \lim \inf x_n = \lim_{n \to \infty} (\inf \{x_n, x_{n+1}, \cdots\}) \in [-\infty, \infty]. \]
2.2. **extinction probability.** In the next example, \( S = \{0, 1, 2, 3, \cdots \} \) is the population size of some single celled organism. Suppose that, in each generation, every cell either:

1. dies with probability \( p_0 \) or
2. divides into \( k \) cells with probability \( p_k \).

Since these are all the possibilities we have:

\[
p_0 + p_1 + p_2 + \cdots = 1.
\]

We have a countable Markov chain where

\[
X_n = \# \text{ cells in generation } n.
\]

2.2.1. **branching process.** This is an example of a branching process. To explain what this means I drew a picture:

\[
\begin{align*}
X_0 &= 1 \\
X_1 &= 2 \\
X_2 &= 3
\end{align*}
\]

Each cell, for example the cell indicated with a circle, starts a branch which looks just like the whole thing. In other words, we have a “fractal” in which every piece looks like the whole. This is a branching process.

2.2.2. **question.** The question is: What is the extinction probability? This is

\[
a := \mathbb{P}(\liminf X_n = 0 \mid X_0 = 1).
\]

This is the probability of extinction when we start with one cell. But what if we start with more than one?

**Lemma 2.2.** \( \mathbb{P}(\liminf X_n = 0 \mid X_0 = k) = a^k \).

**Proof.** I drew a picture to indicate that for each of the \( k \) starting cells, there is a probability \( a \) that that cell and all of its descendants will die out. Assuming independence, the probability that this will happen to each of \( k \) cells is \( a^k \) since we multiply probabilities of independent events. \( \square \)
With this lemma we get a formula for the extinction probability:

\[ a = \mathbb{P}(X_\infty = 0 \mid X_0 = 1) \]

\[ = \sum_{k=0}^{\infty} \mathbb{P}(X_\infty = 0 \mid X_1 = k) \mathbb{P}(X_1 = k \mid X_0 = 1) \]

\[ a = \sum_{k=0}^{\infty} a^k p_k. \]

This is, by definition, the generating function of the sequence \( p_k \).

2.2.3. **Generating functions.** The definition of the generating function \( \phi(s) \) is

\[ \phi(s) := \sum_{k=0}^{\infty} s^k p_k = p_0 + sp_1 + s^2 p_2 + s^3 p_3 + \cdots \]

So, the extinction probability \( a \) is the solution of the equation:

\[ a = \phi(a) \]

We looked at the general properties of the generating function and fed in the specifics of our example. First,

\[ \phi(0) = p_0 > 0. \]

We are going to assume that \( p_0 > 0 \). Otherwise, every cell is immortal and there will be no chance of extinction. Next,

\[ \phi(1) = \sum p_k = p_0 + p_1 + p_2 + \cdots = 1. \]

This is because the probabilities add to 1.

\[ \phi'(s) = \sum_{k=1}^{\infty} ks^{k-1} p_k = p_1 + 2sp_2 + 3s^2 p_3 + \cdots \]

\[ \phi'(1) = \sum kp_k = \mathbb{E}(K) = \mu. \]

This is, by definition, the expected value \( \mu = \mathbb{E}(K) \) of the random variable \( K \) which is the number of offspring of any one cell. It is standard probability notation that the lower case letters are the particular values of upper case letters.

We also need the second derivative:

\[ \phi''(s) = \sum_{k=2}^{\infty} k(k - 1)s^{k-2} p_k = 2p_2 + 6sp_3 + 12s^2 p_4 + \cdots \]

This function is \( \geq 0 \) for all \( s \geq 0 \) and

\[ \phi''(s) > 0 \]
meaning the function is concave up for all \( s > 0 \) if at least one of the numbers \( p_2, p_3, \cdots \) is positive. For example, this must be true if \( \mathbb{E}(K) = 1 \) since \( K \) is sometimes zero: \( p_0 = \mathbb{P}(K = 0) > 0 \), it must also sometimes be greater than 1 to have an average of 1.

**Lemma 2.3.** \( a \) is the smallest positive real number so that \( a = \phi(a) \).

I will give the proof later. First, I want to show what this means.

**Case 1.** Suppose that \( \mu > 1 \). In that case, the slope of the curve \( y = \phi(s) \) at \( s = 1 \) is greater than 1. Since the \( y \)-intercept is \( \phi(0) = p_0 > 0 \), \( \phi(1) = 1 \) and \( \phi \) is concave up, the graph must look like this:

The curves \( y = \phi(s) \) and \( y = s \) must meet at some point \( s = a = \phi(a) \) where \( 0 < a < 1 \). So, there is a nonzero probability of extinction and a nonzero probability of indefinite survival. It could go either way. Someone once told me “May you have a thousand sons.” Even if people have, on the average, a thousand children, if there is a chance of death then there is a chance of extinction of the species.

**Case 2.** Suppose \( \mu < 1 \). In this case the slope is less than 1 at \( s = 1 \) and the graph looks like this:

So, \( a = 1 \). This implies that

\[
\mathbb{P}(\text{extinction} \mid X_0 = k) = a^k = 1.
\]
No matter how large is the population, if the number of children is, on the average, smaller than the number of parents, then the population will almost surely reach extinction. This is not surprising. It is the next case which is really interesting.

Case 3. Suppose $\mu = 1$. In this case the picture is:

The curves $y = \phi(s)$ and $y = s$ must be touching at $s = 1$. This is the only point where these two curves meet. So, $a = 1$ is the only solution of $a = \phi(a)$. So, even if the number of children is, on average, equal to the number of parents, the species will almost surely die out.

2.2.4. computer simulations. Using a random number generator, we looked at some examples of what might happen. When $\mu = 1$, the population sometimes dies out right away:

And sometimes it appears to be surviving:
However, according to the theory, the population will almost surely die out when \( \mu = 1 \). It just does not tell us how long it will take. We will discuss this later.

We did one more simulation with \( \mu = 1.001 \). In this case \( a \approx 99\% \).

When \( X_0 = 10 \) this makes the extinction probability \( a^{10} \approx 90\% \). But 10\% of the time, the species will survive indefinitely. This looks like one of these cases. After 375 generations, the population has reached \( X_{375} = 120 \). So, the extinction probability is now only

\[
a^{120} \approx .99^{120} \approx 30\%.
\]