2.4. Transient, recurrent and null recurrent. [Guest lecture by Alan Haynes. Notes by Connie, Peter, Sam, Xioudi, Zach.] Alan Haynes reviewed the difference between transient and recurrent and defined the notion of null recurrence.

2.4.1. transient and recurrent. Suppose that \( \{X_n\} \) is an irreducible aperiodic Markov chain. Fix \( z \in S \) and define the function \( \alpha : S \to [0,1] \) by
\[
\alpha(x) = \mathbb{P}(X_n = z \text{ for some } n \geq 0 \mid X_0 = x)
\]

Facts:
1. \( \alpha(x) \in [0,1] \)
2. \( \alpha(z) = 1 \)
3. If \( x \neq z \), then
\[
\alpha(x) = \sum_{y \in S} p(x, y) \alpha(y)
\]
\( \Rightarrow \) \( \alpha \) is a right eigenvector of \( P = (p(x, y))_{x,y \in S} \) (the probability transition matrix).

There are two possibilities: \( X_n \) is either transient or recurrent.
4. Transient \( \iff \) the probability of returning to \( z \) (or any other point) an infinite number of times is 0.
\( \iff \alpha(x) < 1 \) for some \( x \in S \)
\( \iff \inf_{x \in S} \alpha(x) = 0 \)
5. Recurrent \( \iff \) the probability of returning to every point an infinite number of times is 1.
\( \iff \alpha(x) = 1 \ \forall x \in S \)

Why is it true that \( \alpha(x) < 1 \) implies \( \inf \alpha(x) = 0 \)? The proof is by contradiction. Suppose not. Then there is some \( \epsilon > 0 \) so that \( \alpha(x) \geq \epsilon \) for all \( x \in S \). This implies that, no matter where you are, you have a probability of at least \( \epsilon \) of returning to \( z \). So, with probability one, you must return to \( z \) an infinite number of times making the chain recurrent.

**Theorem 2.7.** If \( \tilde{\alpha} : S \to [0,1] \) satisfies
\[
\tilde{\alpha}(x) = \sum_{y \in S \setminus z} p(x, y) \tilde{\alpha}(y) \text{ for all } x \in S \setminus z
\]
then \( \alpha(x) \leq \tilde{\alpha}(x) \) for all \( x \in S \).
To show that a Markov chain is transient, you should find a solution $\hat{\alpha}(x)$ of the equation so that $\hat{\alpha}(x) < 1$ at some point $x \in S$.

Example: Take the random walk on $\{0, 1, 2, \cdots\}$ with partially reflecting wall at 0. This means that, at 0, you either stay at 0 or go to 1. Choose some $p$ in the open interval $(0, 1)$ and let $q = 1 - p$. If $n \geq 1$, the transition probabilities are given by

$$p(n, n - 1) = p \quad \text{(going to the left)}$$

$$p(n, n + 1) = q \quad \text{(going to the right)}$$

At $n = 0$ you can’t go left. The probabilities are given by

$$p(0, 0) = p, \quad p(0, 1) = q.$$  

Question: Is this transient or recurrent? We need to choose a point $z$ and use the theorem.

Choose $z = 0$. Then find a function $\hat{\alpha}$ that satisfies the equation in the theorem:

$$\hat{\alpha}(n) = \sum_{m \in \mathbb{N}} p(n, m)\hat{\alpha}(m) \quad \text{for } n \geq 1$$

Since $p(n, m) = 0$ unless $m = n + 1, n - 1$, we can simplify this to:

$$\hat{\alpha}(n) = p\hat{\alpha}(n - 1) + q\hat{\alpha}(n + 1)$$

This is a homogeneous linear difference equation with constant coefficients. So,

$$\hat{\alpha}(n) = C_1 r^n + C_2 s^n$$

where $C_1, C_2 \in \mathbb{R}$ and $r, s$ are the roots of the equation

$$x = p + qx^2$$

If you use the quadratic equation you get:

$$r = \frac{1 - \sqrt{1 - 4pq}}{2q}, \quad s = \frac{1 + \sqrt{1 - 4pq}}{2q}$$

If you use the fact that $p + q = 1$ you get the solution:

$$r, s = \frac{p}{q}, 1 \quad \text{(The smaller number is } r.\})$$

To get this you need to factor the equation:

$$x = (p + q)x = p + qx^2$$

$$p(x - 1) = q(x^2 - x)$$

This is true if $x = 1$ or, when $x \neq 1$ you can factor out the term $x - 1$ to get:

$$p = qx.$$
Since the actual probability is $\alpha(n) \leq \hat{\alpha}(n)$, we want $\hat{\alpha}(n) < 1$. This happens when one of the roots $r, s$ is less than 1.

**Case 1:** $p = \frac{1}{2}$. Then $q = \frac{1}{2}$. The roots are $r = s = 1$. Since

$$\hat{\alpha}(0) = C_1r^0 + C_2s^0 = 1$$

we have

$$\hat{\alpha}(n) = C_1r^n + C_2s^n = C_1 + C_2 = 1.$$ 

So, the only solution of the equation is $\hat{\alpha}(n) = 1$. This implies the chain is recurrent. (In fact, it is *null recurrent* as explained below.)

**Case 2:** $p < \frac{1}{2}$. Then $p = \frac{1}{2} - \epsilon, q = \frac{1}{2} + \epsilon, pq = \frac{1}{4} - \epsilon^2, 1 - 4pq = 4\epsilon^2$.

$$r = \frac{1 - \sqrt{1 - 4pq}}{2q} = \frac{1 - 2\epsilon}{1 + 2\epsilon} = \frac{p}{q} < 1.$$ 

Then, one of the solutions is:

$$\hat{\alpha}(n) = r^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

This implies that the chain is transient.

Note: We only need one $\hat{\alpha}(n) < 1$ to prove transience.

**Case 3:** $p > \frac{1}{2}$. Then $p = \frac{1}{2} + \epsilon, q = \frac{1}{2} - \epsilon, pq = \frac{1}{4} - \epsilon^2$.

$$r = \frac{1 - \sqrt{1 - 4pq}}{2q} = \frac{1 - 2\epsilon}{1 - 2\epsilon} = 1.$$ 

The other solution is

$$s = \frac{1 + \sqrt{1 - 4pq}}{2q} = \frac{1 + 2\epsilon}{1 - 2\epsilon} = \frac{p}{q} > 1.$$ 

This implies the chain is recurrent. (In fact, it is *positive recurrent.*)
2.4.2. *null recurrence.*

**Definition 2.8.** An irreducible aperiodic chain \( \{X_n\} \) is called *null recurrent* if it is recurrent and

\[
\lim_{n \to \infty} p_n(x, y) = 0 \quad \forall x, y \in S.
\]

If \( \{X_n\} \) is recurrent but not null recurrent then it is called *positive recurrent.*

How do you tell the difference?

**Theorem 2.9.** If \( \{X_n\} \) is positive recurrent then, for every \( x, y \in S \),

\[
\lim_{n \to \infty} p_n(x, y) = \pi(y) > 0
\]

where \( \pi : S \to [0, 1] \) is the invariant probability distribution.

Recall the definition of invariant distribution:

1. (probability distribution) \( \sum_{x \in S} \pi(x) = 1 \)
2. (invariant) \( \pi P = \pi : \]

\[
\pi(y) = \sum_{x \in S} \pi(x)p(x, y).
\]

I.e., \( \pi \) is a left eigenvector of \( P \).

**Corollary 2.10.** \( \{X_n\} \) is positive recurrent if and only if it has an invariant distribution.