8.4. Brownian motion in $\mathbb{R}^d$ and the heat equation. The heat equation is a partial differential equation. We are going to convert it into a probabilistic equation by reversing time. Then we can using stopping time.

8.4.1. Definition. 

Definition 8.12. $d$-dimensional Brownian motion with drift $\mu = 0 \in \mathbb{R}^d$ and variance $\sigma^2$ is a vector valued stochastic process

$$X_t \in \mathbb{R}^d, \quad t \in [0, \infty)$$

$$X_t = (X_t^1, X_t^2, \cdots, X_t^d)$$

so that

1. $X_t$ is continuous
2. The increments $X_{t_i} - X_{s_i}$ of $X_t$ on disjoint time intervals $(s_i, t_i]$ are independent.
3. Each coordinate of the increment is normal with the same variance:

$$X_{t_i}^j - X_{s_i}^j \sim N(0, \sigma^2(t - s))$$

and they are independent.

This implies that the density function of $X_t - X_s$ is a product of normal density functions:

$$f_{t-s}(x) = f_{t-s}(x_1) \cdots f_{t-s}(x_d) = \prod_{i=1}^d \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} e^{-x_i^2/2\sigma^2(t-s)}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} e^{-||x||^2/2\sigma^2(t-s)}$$

since $\prod e^{\text{whatever}} = e^{\sum \text{whatever}}$ and

$$\sum_{i=1}^d x_i^2 = ||x||^2.$$

Since this is the density of the increment $X_t - X_s$, it gives the transition “matrix”

$$p_{\Delta t}(x, y) = f_{\Delta t}(y - x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} e^{-||y-x||^2/2\sigma^2\Delta t}$$

The point is that $||y - x|| = ||x - y||$. So,

$$p_{\Delta t}(x, y) = p_{\Delta t}(y, x).$$

In other words, Brownian motion (with zero drift) is a symmetric process. When you reverse time, it is the same. (It is also obvious that if there is a drift $\mu$, the time reversed process will have drift $-\mu$.)
8.4.2. diffusion (the heat equation). If we have a large number of particles moving independently according to Brownian motion then the density of particles at time \( t \) becomes a deterministic process called diffusion. It satisfies a differential equation called the heat equation. When we reverse time, we will get a probabilistic version of this equation called the “backward equation.”

Let \( f(x) \) be the density of particles (or heat) at position \( x \) at time \( t \). Then we have the Chapman-Kolmogorov equation, also called the forward equation:

\[
 f_{t+\Delta t}(y) = \int_{\mathbb{R}^d} f_t(x)p_{\Delta t}(x, y) \, d\mathbf{x}
\]

But, \( p_{\Delta t}(x, y) = p_{\Delta t}(y, x) \) since Brownian motion is symmetric when \( \mu = 0 \). So,

\[
 f_{t+\Delta t}(y) = \int_{\mathbb{R}^d} f_t(x)p_{\Delta t}(y, x) \, dx
\]

Since equations remain true when you change the names of the variables, this equation will still hold if I switch \( x \leftrightarrow y \). This gives the backward equation:

\[
 f_{t+\Delta t}(x) = \int_{\mathbb{R}^d} f_t(y)p_{\Delta t}(x, y) \, dy
\]

The RHS is an expected value since it is the sum of \( f(y) \) times its probability. Since \( x \) moves to \( y \) in time \( \Delta t \), \( y = X_{t+\Delta t} \).

\[
 \int_{\mathbb{R}^d} f_t(y)p_{\Delta t}(x, y) \, dy = \mathbb{E}_t^x (f_t(X_{t+\Delta t})) = \mathbb{E}(f_t(X_{t+\Delta t}) \mid X_t = x)
\]

Where \( \mathbb{E}_t^x \) means expectation is conditional on \( X_t = x \). In words:

**Future density at the present location \( x \)**

= expected value of the present density at the future location \( y \)

using the following interpretation of “present” and “future”

<table>
<thead>
<tr>
<th>time</th>
<th>location</th>
</tr>
</thead>
<tbody>
<tr>
<td>present</td>
<td>( t )</td>
</tr>
<tr>
<td>future</td>
<td>( t + \Delta t )</td>
</tr>
</tbody>
</table>
8.4.3. Calculate $\frac{\partial}{\partial t} f_t$. I want to calculate

$$\frac{\partial}{\partial t} f_t(x) = \lim_{\Delta t \to 0} \frac{f_t(x) - f_t(x)}{\Delta t}.$$ 

Using the backward equation, this is

$$= \lim_{\Delta t \to 0} \frac{\mathbb{E}^x_t(f_t(X_{t+\Delta t})) - f_t(x)}{\Delta t}$$

To figure this out I used the Taylor series. Here it is when $d = 1$.

$$f_t(X_{t+\Delta t}) = f_t(X_t) + f_t'(X_t)\Delta X + \frac{1}{2} f_t''(X_t)((\Delta X)^2) + O((\Delta X)^3)$$

Here $f_t' = \frac{\partial}{\partial x} f_t$. The increment in $X$ is

$$\Delta X = X_{t+\Delta t} - X_t \sim N(0, \sigma^2 \Delta t)$$

This means that

$$\mathbb{E}((\Delta X)^2) = \sigma^2 \Delta t.$$ 

In other words, $(\Delta X)^2$ is expected to be on the order of $\Delta t$. So, $(\Delta X)^3$ is on the order of $(\Delta t)^{3/2}$. So,

$$\frac{\mathbb{E}(\epsilon)}{\Delta t} = \frac{\mathbb{E}(O((\Delta X)^3))}{\Delta t} \to 0 \quad \text{as } \Delta t \to 0$$

Taking expected value and substituting $X_t = x$ we get:

$$\mathbb{E}_x^t(f_t(X_{t+\Delta t})) - f_t(x) = f_t'(x) \mathbb{E}((\Delta X) + \frac{1}{2} f_t''(x)\mathbb{E}_x^t((\Delta X)^2) + \mathbb{E}(\epsilon)$$

$$= \frac{1}{2} f_t''(x)\sigma^2 \Delta t + \mathbb{E}(\epsilon)$$

$$\frac{\mathbb{E}_x^t(f_t(X_{t+\Delta t})) - f_t(x)}{\Delta t} = \frac{1}{2} f_t''(x)\sigma^2 + \frac{\mathbb{E}(\epsilon)}{\Delta t}$$

So,

$$\frac{\partial}{\partial t} f_t(x) = \frac{\sigma^2}{2} f_t''(x)$$

In higher dimensions we get the following

$$\frac{\partial}{\partial t} f_t(x) = \frac{\sigma^2}{2} \Delta f_t(x)$$

where $\Delta$ is the Laplacian:

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$
This follows from the multivariable Taylor series:

\[ f_t(X_{t+\Delta t}) = f_t(X_t) + \sum_i \frac{\partial f_t(X_t)}{\partial x_i} \Delta X^i_t + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f_t(X_t)}{\partial x_i \partial x_j} \Delta X^i_t \Delta X^j_t + \epsilon \]

Since \( \mathbb{E}(\Delta X^i_t) = -\mu_i = 0 \), the \( \Sigma_i \) terms have expected value zero and the \( \Sigma_{i,j} \) terms also have zero expected value when \( i \neq j \). This leaves the \( \Sigma_{i,i} \) terms which give the Laplacian. I pointed out in class that \( \mathbb{E}(\Delta X^i_t) = -\mu_i \) because we are using the backward equation.

8.4.4. boundary values. Now we want to solve the boundary valued problem, or at least convert it into a probability equation. Suppose we have a bounded region \( B \) and we heat up the boundary \( \partial B \).

Let

\[ f(x) = \text{current temperature at } x \in B \]
\[ g(y) = \text{current temperature at } y \in \partial B \]

Suppose the \( g(y) \) is fixed for all \( y \in \partial B \). This is the heating element on the outside of your oven. The point \( x \) is in the inside of your oven. The temperature \( f(x) \) is changing according to the heat equation:

\[ \frac{\partial f}{\partial t} = \frac{\sigma^2}{2} \Delta f. \]

We want to calculate \( u(t, x) = f_t(x) \), the temperature at time \( t \). We also want the equilibrium temperature \( v(x) = f_\infty(x) \). When the oven has been on for a while it stabilizes and

\[ \frac{\partial}{\partial t} f_\infty(x) = 0 \]

which forces

\[ \Delta f_\infty(x) = 0 \]

If we use the backward equation we can use the stopping time \( T = \) the first time you hit \( \partial B \). Taking it to be a stopping time means that
the boundary is “sticky” like flypaper. The particle $x$ bounces around inside the region $B$ until it hits the boundary $\partial B$ and then it stops. (You can choose your stopping time to be anything that you want.) Then the backward equation, using OST, is:

$$v(t, x) = \mathbb{E}^x(g(X_T)I(T \leq t) + f(X_T)I(t < T))$$

Here $I(T \leq t)$ is the indicator function for the event that $T \leq t$. Multiplication by this indicator function is the same as the condition “if $T \leq t$.” The equilibrium temperature is given by

$$f_\infty(x) = v(x) = \mathbb{E}^x(g(X_T))$$

This is an equation we studied before. $v(x)$ is the value function. It gives your expected payoff if you start at $x$ and use the optimal strategy. $g(x)$ is the payoff function. $X_T$ is the place that you will eventually stop if you use your optimal strategy which is the formula for the stopping time $T$.

I gave one really simple example to illustrate this concept.

**Example 8.13.** You give a professional gambler $\$x$ and send him to a casino to play until he loses (when he has $\$0$) or wins (by getting $y = \$10^3$). The gambler gets a fee of $\$a$ if he loses and $\$b$ if he wins. The question is: What is his expected payoff?

$T$ = stopping time is the first time that $X_T = 0$ or $y$. We have $B = [0, y]$ with boundary $\partial B = \{0, y\}$ and boundary values:

$$g(0) = a, \quad g(y) = b.$$ 

The expected payoff, starting at $x$, is

$$v(x) = \mathbb{E}^x(g(X_T)) = a\mathbb{P}^x(X_T = 0) + b\mathbb{P}^x(X_T = y).$$

But, $\mathbb{E}^x(X_T) = X_0 = x$ by the Optimal Sampling Theorem. This is:

$$y\mathbb{P}^x(X_T = y) = x$$

making $\mathbb{P}^x(X_T = y) = x/y$ and

$$\mathbb{P}^x(X_T = 0) = 1 - \frac{x}{y}.$$ 

Therefore,

$$v(x) = a \left(1 - \frac{x}{y}\right) + b \frac{x}{y}.$$ 

This is a linear function which is equal to $a$ at $x = 0$ and $v(y) = b$. While this was fun, this is an example where the boring analytic method is actually much more efficient: The heat equation says

$$\Delta v(x) = v''(x) = 0$$
Figure 1. \( v(x) \) is the convex function given by the convex hull of the points \( (0, a), (y, b) \) which are the given payoffs.

In other words, the derivative \( v'(x) \) is constant and \( v(x) \) is a straight line function. With the given values \( v(0) = a, v(y) = b \) we get the answer right away.