2. Answers to Worksheet 2

\[ P = \begin{pmatrix}
.8 & 0 & .2 & 0 & 0 \\
0 & .5 & 0 & 0 & .5 \\
.3 & 0 & .7 & 0 & 0 \\
.2 & 0 & 0 & .6 & .2 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix} \]

(1) Calculate the rank of \( P - I_5 \) by column reduction.

\[ P - I = \begin{pmatrix}
-.2 & 0 & .2 & 0 & 0 \\
0 & -.5 & 0 & 0 & .5 \\
.3 & 0 & -.3 & 0 & 0 \\
.2 & 0 & 0 & -.4 & .2 \\
0 & 1 & 0 & 0 & -1
\end{pmatrix} \]

The first step is to clear the last column. This is a column operation given by adding the first 4 columns to the last column:

\[ \begin{pmatrix}
-.2 & 0 & .2 & 0 & 0 \\
0 & -.5 & 0 & 0 & .5 \\
.3 & 0 & -.3 & 0 & 0 \\
.2 & 0 & 0 & -.4 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix} \]

Divide the fourth column by -.4 and use it to clear the fourth row:

\[ \begin{pmatrix}
-.2 & 0 & .2 & 0 & 0 \\
0 & -.5 & 0 & 0 & 0 \\
.3 & 0 & -.3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix} \]

Now add the first column to the 3rd column:

\[ \begin{pmatrix}
-.2 & 0 & 0 & 0 & 0 \\
0 & -.5 & 0 & 0 & 0 \\
.3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix} \]

This is good enough because each row has only one nonzero entry.
(2) How many recurrent classes does this Markov chain have?

The rank of $P - I$ is equal to 3 (the number of nonzero columns in the reduced form). Therefore, the nullity is $5 - 3 = 2$. (This is the number of zero columns in the reduced form.) So, there are exactly 2 recurrent classes.

(3) Find the left null vectors of $P - I_5$. Normalize to get the basic invariant distributions.

The left null vectors are:

$$(3, 0, 2, 0, 0), \quad (0, 2, 0, 0, 1).$$

These are the solutions of the equations

$$(−.2)x_1 + .3x_3 = 0$$
$$(−.5)x_2 + (1)x_5 = 0$$
$$x_4 = 0$$

which come from the matrix equation:

$$
\begin{pmatrix}
-0.2 & 0 & 0 & 0 & 0 \\
0 & -0.5 & 0 & 0 & 0 \\
0.3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
= \begin{pmatrix} 0, 0, 0, 0 \end{pmatrix}.
$$

When you normalized these two vectors, you get the basic invariant probability distributions:

$$\pi_1 = \left( \frac{3}{5}, 0, \frac{2}{5}, 0, 0 \right), \quad \pi_2 = \left( 0, \frac{2}{5}, 0, 0, \frac{1}{5} \right)$$

(4) What are the supports of these vectors? These are the recurrent classes.

The supports are the coordinates which are nonzero. These are:

$$\text{supp}(\pi_1) = \{1, 3\} = R_1, \quad \text{supp}(\pi_2) = \{2, 5\} = R_2.$$

(5) Find all invariant distributions $\pi$.

These are given by the formula:

$$\pi = t\pi_1 + (1-t)\pi_2 = \left( \frac{3t}{5}, \frac{2(1-t)}{3}, \frac{2t}{5}, 0, \frac{1-t}{3} \right)$$

where $0 \leq t \leq 1$.

(6) If the initial distribution is

$$\alpha = \left( 0, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, 0 \right),$$

what is the invariant distribution

$$\alpha P^\infty := \lim_{n \to \infty} \alpha P^n \ ?$$

To answer this we need to determine $t$, the probability of ending up in recurrent state $R_1$. Since 2 and 3 are in $R_2, R_1$ respectively, we start in each of those recurrent classes with probability $1/3$. For the remaining $1/3$rd of the time we start in the transient state 4. We need to look at the 4th row of the original matrix $P$ to see what happens in that case:

$$(.2, 0, 0, .6, .2).$$
This show a loop at 4. When you escape that loop you go with equal probability to 1 and 5 which are in the recurrent classes $R_1, R_2$. So, you end up in $R_1, R_2$ later with probability $1/6, 1/6$ resp. In total, the probability is $1/2, 1/2$ that you end up in $R_1, R_2$. So,

$$t = \mathbb{P}(X_n \in R_1 \text{ for large } n) = \frac{1}{2}.$$ 

So,

$$\alpha P^{\infty} = \pi = \left( \frac{3}{10}, \frac{1}{3}, \frac{1}{5}, 0, \frac{1}{6} \right).$$

(7) Renumber the states and put the matrix $P$ into canonical form.

The recurrent states should come first. To avoid confusion, I will use letters to indicate the reordering:

$$a = 1, b = 3, c = 2, d = 5, e = 4.$$ 

Then the matrix $P$ becomes:

$$
\begin{pmatrix}
P_1 & 0 & 0 \\
0 & P_2 & 0 \\
S_1 & S_2 & Q
\end{pmatrix}
= 
\begin{pmatrix}
.8 & .2 & 0 & 0 & 0 \\
.3 & .7 & 0 & 0 & 0 \\
0 & 0 & .5 & .5 & 0 \\
0 & 0 & 1 & 0 & 0 \\
.2 & 0 & 0 & .2 & .6
\end{pmatrix}
$$