0.1. Abstract. This is joint work with Gordana Todorov. Let $R = K[[t]]$ where $K$ is any field. Given a “recurrent” cyclic poset $X$ and “admissible automorphism” $\phi$, we construct an $R$-linear Frobenius category $\mathcal{F}_\phi(X)$. I will go over the definition of a Frobenius category and indicate why our construction satisfies each condition. By a well-known result of Happel, the stable category $\mathcal{C}_\phi(X)$ will be a triangulated category over $K$. In each example in the chart below, $\mathcal{C}_\phi(X)$ will be a cluster category:

<table>
<thead>
<tr>
<th>cyclic poset</th>
<th>automorphism</th>
<th>cluster category</th>
<th>comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$\phi$</td>
<td>$C_\phi(X)$</td>
<td></td>
</tr>
<tr>
<td>$Z_n$</td>
<td>$\phi(i) = i + 1$</td>
<td>$\mathcal{C}(A_{n-3})$</td>
<td>2-CY</td>
</tr>
<tr>
<td>$1 &lt; 2 &lt; \cdots &lt; n &lt; \sigma 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>$\phi(i) = i + 1$</td>
<td>$\mathcal{C}(A_\infty)$</td>
<td>2-CY</td>
</tr>
<tr>
<td>(with cyclic order)</td>
<td></td>
<td>infinity-gon</td>
<td></td>
</tr>
<tr>
<td>$S^1$</td>
<td>$id$</td>
<td>$\mathcal{C}$</td>
<td>continuous cluster category $Y[1] \cong Y$</td>
</tr>
<tr>
<td>$S^1 \ast \mathbb{Z}$</td>
<td>$id$</td>
<td>$\tilde{\mathcal{C}}$</td>
<td>not 2-CY ($Y[1] \cong Y$)</td>
</tr>
<tr>
<td>$\phi(x, i) = (x, i + 1)$</td>
<td></td>
<td>$\tilde{\mathcal{C}}'$</td>
<td>2-CY</td>
</tr>
<tr>
<td>$Z_m \ast \mathbb{Z}$</td>
<td>$\phi(i, j) = (i + 1, j)$</td>
<td>$\mathcal{C}$</td>
<td>contains $(m + 1)$-CY</td>
</tr>
<tr>
<td>$\phi(m, j) = (1, j + 1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{P}(1)/3 \mathbb{Z} \ast \mathbb{Z}$</td>
<td>$\phi^3(x, i) = (x, i + 1)$</td>
<td>$(3$-cluster category of type $A_\infty)^3$</td>
<td>4-CY</td>
</tr>
</tbody>
</table>

I will go over some of the easier examples of this construction. CY means Calabi-Yau.
1.1. **Cyclic poset.** is same as periodic poset $\tilde{X}$. i.e. $\exists$ poset automorphism $\sigma : \tilde{X} \to \tilde{X}$ so that $x < \sigma x$ for all $x$. Also:

- $(\forall x, y \in \tilde{X})$ $x \leq \sigma^j y$ for some $j \in \mathbb{Z}$.

1. Z$_n$: $\tilde{X} = \mathbb{Z}$, $\sigma(x) = x + n$ ($n$ fixed).
2. $X = S^1$. Then $\tilde{X} = \mathbb{R}$ with $\sigma(x) = x + 2\pi$.
3. $\tilde{X} \ast \mathbb{Z}$ means $\tilde{X} \times \mathbb{Z}$ with lexicographic order (from van Roosmalen).

Let $X$ = set of $\sigma$ orbits. How to describe cyclic poset structure just in terms of $X$?

Following, van Roosmalen 1011.6077, p.10 and Drinfeld 0304064, p.5, (who refers to Besser and Greyson), this structure is equivalent to an $\mathbb{N}$-category structure on $X$.

**Definition 1.1.1.** An $\mathbb{N}$-category is a category $\mathcal{X}$ with the property that the additive monoid $\mathbb{N}$ acts freely on every Hom set

$$\mathbb{N} \times \mathcal{X}(x, y) \to \mathcal{X}(x, y)$$

so that composition satisfies:

$$nf \circ mg = (n + m)fg : x \to z$$

(Acting freely means Hom sets are disjoint unions of copies of $\mathbb{N}$: $\mathcal{X}(x, y) = \bigsqcup \mathbb{N}f_i$.)

**Proposition 1.1.2.** A cyclic poset structure on a set $X$ is the same as an $\mathbb{N}$-category $\mathcal{X}$ with object set $X$ so that every Hom set $\mathcal{X}(x, y)$ is freely generated by one morphism $f_{xy}$.

So, given three objects, $x, y, z \in X$, we have

$$f_{yz}f_{xy} = nf_{xz}$$

for some $n \in \mathbb{N}$.

1.2. **Linearized cyclic poset.** We write: $\mathcal{X} = (X, c)$.

**Definition 1.2.1.** For any field $\mathbb{k}$, the (completed) linearization $\widehat{\mathcal{X}}$ of $\mathcal{X}$ is defined to be the category with object set $X$ and morphism sets

$$\widehat{\mathcal{X}}(x, y) = \mathbb{k}^{\mathcal{X}(x, y)} \cong \mathbb{k}[t]$$

composition is given by

$$(rf_{yz}) \circ (sf_{xy}) = rst^n f_{xz}$$

for any $r, s \in R := \mathbb{k}[t]$ where $n$ is given by (1.1).

This is an $R$-category: Hom sets are $R$-modules and composition is $R$-bilinear.
**Definition 1.2.2.** A *representation* of $\mathcal{X}$ is defined to be an $R$-linear functor

$$M : \widehat{k}\mathcal{X} \to R\text{-mod}$$

**Definition 1.2.3.** Let $\mathcal{P}(\mathcal{X})$ be the category of all finitely generated projective representations of $\mathcal{X}$.

**Proposition 1.2.4 (Yoneda).** $\mathcal{P}(X) \cong \text{add}\widehat{k}\mathcal{X}^{op}$

Let $P_x$ be the projective representation of $X$ generated at the point $x \in X$.

1.3. Frobenius category.

**Definition 1.3.1.** Let $\mathcal{F}(X)$ denote the category of all pairs $(P, e)$ where $P \in \mathcal{P}(X)$ and $e : P \to P$ so that $e^2 = \cdot t$ (mult by $t$). Morphism $f : (P, e) \to (Q, e)$ are maps $f : P \to Q$ so that $ef = fe$.

**Lemma 1.3.2.** The functor $G : \mathcal{P}(X) \to \mathcal{F}(X)$ given by

$$GP := \left( P \oplus P, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right) : \begin{array}{ccc} P & \xrightarrow{t} & P \\ \downarrow{id} & & \downarrow{id} \\ P & \xleftarrow{t} & P \end{array}$$

is both left and right adjoint to the forgetful functor $F : \mathcal{F}(X) \to \mathcal{P}(X)$.

**Theorem 1.3.3.** For any cyclic poset $X$, $\mathcal{F}(X)$ is a Frobenius category where a sequence

$$(A, e) \mapsto (B, e) \mapsto (C, e)$$

is defined to be exact in $\mathcal{F}(X)$ if $A \mapsto B \mapsto C$ is (split) exact in $\mathcal{P}(X)$. $GP$ are the projective injective objects.

**Proof.** We can easily verify each step in the definition of a Frobenius category, namely, a Frobenius category is an exact category which has enough projectives so that all projective objects are injective. (E.g., $kG\text{-mod}$ for any finite group $G$.) An exact category is an additive category with a collection of exact sequences

$$\mathcal{E} = \{ A \mapsto f B \mapsto g C \}$$

so that

1. $A = \ker g$ and $C = \coker f$
2. $0 \mapsto 0 \mapsto 0 \in \mathcal{E}$
3. Given $A \mapsto f_1, B \mapsto \coker f_1, B \mapsto f_2, C \mapsto \coker f_2$ in $\mathcal{E}$, then

$$A \xrightarrow{f_2 f_1} C \mapsto \coker f_2 f_1 \in \mathcal{E}$$
4. The pushout of any $A \mapsto B \mapsto C \in \mathcal{E}$ along any $f : A \to A'$ is in $\mathcal{E}$.
5. Dually for pull backs along $C' \to C$.

Finally, $GP$ is projective since $G$ is left adjoint to $F$ and $GP$ is injective because $G$ is right adjoint to $F$. So, $\mathcal{F}(\mathcal{X})$ is Frobenius. \[\square\]
1.4. **Twisted version.** An automorphism \( \phi \) of \( X \) is admissible if:

\[
x \leq \phi(x) \leq \phi^2(x) \leq \sigma x
\]

for all \( x \in \tilde{X} \). In \( \tilde{\mathcal{X}} \) this gives

\[
P_x \xrightarrow{\eta_P} \phi P_x = P_{\phi(x)} \xrightarrow{\xi_P} P_x
\]

giving natural transformations

\[
P \xrightarrow{\eta_P} \phi P \xrightarrow{\xi_P} P
\]

**Definition 1.4.1.** Let \( \mathcal{F}_\phi(X) \) be the full subcategory of \( \mathcal{F}(X) \) of all \( (P, e) \) where \( e \) factors through \( \eta_P : P \to \phi P \).

**Theorem 1.4.2.** \( \mathcal{F}_\phi(X) \) is a Frobenius category with projective-injective objects

\[
G_\phi P := \left( P \oplus \phi P, \begin{bmatrix} 0 & \xi_P \\ \eta_P & 0 \end{bmatrix} \right) : \quad P \xleftarrow{\eta_P} \phi P
\]

2. **Cluster categories**

**Definition 2.0.3.** The stable category \( \mathcal{F} \) of a Frobenius category \( \mathcal{F} \) has the same set of objects as \( \mathcal{F} \) with morphism sets:

\[
\mathcal{F}(A, B) = \frac{\mathcal{F}(A, B)}{A \to P \to B, P \text{ proj-inj}}
\]

**Theorem 2.0.4** (Happel). The stable category of a Frobenius category is triangulated.

**Definition 2.0.5.** Let \( \mathcal{C}(X) = \mathcal{F}(X) \) and \( \mathcal{C}_\phi(X) = \mathcal{F}_\phi(X) \).

**Theorem 2.0.6.** In all examples on page 1, \( \mathcal{F}(X) \) is Krull-Schmidt \( R \)-category with indecomposable objects:

\[
M(x, y) := \left( P_x \oplus P_y, \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix} \right) : \quad P_x \xleftarrow{\alpha} \phi \beta P_y
\]

with \( x, y \in X \).

**Corollary 2.0.7.** \( \mathcal{C}_\phi(X) \) is Krull-Schmidt \( k \)-category with indecomposable objects \( M(x, y) \) where \( y \neq \phi x, \phi^{-1}x \).

**Remark 2.0.8.** Cluster categories were first constructed by Buan-Marsh-Reineke-Reiten-Todorov (0402054) as orbit categories. This construction is an alternate construction in type \( A \).
2.1. **Example** \( X = \mathbb{Z}_n, \phi(i) = i + 1 \). The cyclic poset has \( n \) elements in a circle. Indecomposable objects are \( M(x, y) \) where \( x, y \) are at least two steps apart (because \( M(i, i + 1) \) is projective-injective). This is the well-known CCS model \((0401316)\) for the cluster category of type \( A_{n-3} \). (But they did not give the triangulated structure of the category.)

2.2. **Example** \( X = \mathbb{Z}, \phi(i) = i + 1 \). Indecomposable objects are \( M(x, y) \) where \( x, y \) are at least two steps apart (because \( M(i, i + 1) \) is projective-injective). This is the well-known CCS model \((0902.4125)\).

2.3. **Example** \( S^1 \) with \( \phi = \text{id} \). The objects are \( M(x, y) \) where \( x, y \) are distinct points on the circle. This is the continuous cluster category \((1209.1879)\).

2.4. **Example** \( X = \mathbb{Z}_5 \ast \mathbb{Z} \).

\[
\phi(x, i) = \begin{cases} 
(x + 1, i) & \text{if } 1 \leq x < 5 \\
(1, i + 1) & \text{if } x = 5 
\end{cases}
\]

Then \( C_\phi(X) \) is 6-CY.

**Theorem 2.4.1.** Maximal compatible sets of 6 rigid objects correspond to 2-periodic partitions of the doubled \( \infty \)-gon into 7-gons (except for the one in the middle).

Example of a maximal compatible set of 6-rigid objects in \( C_\phi(\mathbb{Z}_5 \ast \mathbb{Z}) \). \( M(x, y) \) is arc from \( x \) to \( y \) (horizontal if standard, vertical if nonstandard). Compatible arcs do not cross. There is 8-gon in center. Other regions have 7 sides.

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
\end{array}
\]

\[
Y_2 \quad 7\text{-gon} \quad Y_1 \quad 8\text{-gon} \quad Y_1 \quad 7\text{-gon} \quad Y_2
\]

\[
\begin{array}{cccccccccccc}
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Standard: \( X_1 = M(A_0, B_1) \) (horizontal).
\( Y_1 = M(C_1, E_{-1}), Y_2 = M(A_{-1}, D_1) \) are nonstandard but \((m + 1)\)-rigid (vertical).

Notation: \((1, j) = A_j, (2, j) = B_j, \) etc.