Math 211a, Fall 2004, Homework # 4
Van der Waerden’s Theorem and Multiple Recurrence

1. [Brin & Stuck, Exercises (2.8.1–3), 2.8.4].

2. Give an example of a finite partition $\mathbb{Z} = \bigcup_{i=1}^{r} A_i$ such that none of the sets $A_i$ contains an infinite arithmetic progression.

3. Say that $A \subset \mathbb{Z}$ is $AP$-rich if it contains arbitrary long arithmetic progressions. Prove that syndetic sets are $AP$-rich.

4. Let $A = \bigcup_{i=1}^{r} A_i$ be a finite partition of an $AP$-rich set $A \subset \mathbb{Z}$; show that one of the sets $A_i$ is $AP$-rich. (This is clearly an equivalent version of Van der Waerden’s Theorem.)

5. Construct explicitly a sequence $\omega \in \{0, 1\}^\mathbb{N}$ which is recurrent both for the left shift $\sigma$ and for its square $\sigma^2$, but is not doubly recurrent for $\sigma$.

6. Let $(X, f)$ be a topological dynamical system, and let a closed $D \subset X$ be homogeneous and weakly recurrent for $f$ (that is, $\forall \varepsilon > 0 \ \exists x, y \in D$ and $n \in \mathbb{N}$ with $d(x, f^n(y)) < \varepsilon$). Show that $D$ contains a dense set of recurrent points. (Equivalently, if $G = \langle f_1, \ldots, f_\ell \rangle$ is commutative and acts minimally on $X$, then there is a dense set of multiply recurrent points.)