In all of the problems below, a dynamical system \((X, f)\) is a continuous map \(f\) of a compact metric space \(X\) with distance function \(d\).

1. [Brin & Stuck, Exercises (2.2.1–3) and (2.3.1–4)].

2. [Brin & Stuck, Exercise 2.2.6]. (Hint: assume that there exists a proper minimal subset, and consider its intersections with fibers...)

3. Suppose that \(X\) has no isolated points and \(f\) is topologically transitive. Prove that there exists a dense \(G_\delta\) set of points with dense orbits (\(G_\delta\) stands for countable intersections of open sets).

Recall that \(A \subset \mathbb{N}\) is called a set of topological recurrence (abbreviated by STR) if for any dynamical system \((X, f)\) and any \(\varepsilon > 0\) there exist \(x \in X\) and \(n \in A\) with \(d(x, f^n(x)) < \varepsilon\). (Equivalently, for any finite coloring \(\mathbb{N} = \bigcup_{i=1}^r A_i\) there exists a color \(c \in \{1, \ldots, r\}\) such that the difference between two elements of \(A_c\) is in \(A\).)

4. Let \(A = \bigcup_{i=1}^r A_i\) be a finite partition of STR \(A \subset \mathbb{N}\); show that at least one of the sets \(A_i\) is STR.

5. Prove that any STR contains a disjoint union of infinitely many STRs.

6. Let \(\{a_n\} \subset \mathbb{N}\) be infinite. Prove that the following sets are STR:
   (a) the difference set \(\{a_m - a_n \mid m, n \in \mathbb{N}, a_m > a_n\}\);
   (b) the set of the form \(\bigcup_{n=1}^{\infty} \{a_n, \ldots, na_n\}\).

7. Let \(A\) be STR. Prove that for any \(k \in \mathbb{N}\), (a) \(A \setminus \{1, \ldots, k\}\), (b) \(kA\) and (c) \(\frac{1}{k} A \overset{\text{def}}{=} \{n \in \mathbb{N} \mid kn \in A\}\) are STRs.

8. Prove or disprove: any STR contains (a) \(a, b, c\) with \(a + b = c\); (b) an arithmetic progression of length 3.

9. Recall that an increasing sequence \(\{a_n\} \subset \mathbb{N}\) is called lacunary if

\[
\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} > 1.
\]

Prove that a lacunary sequence cannot be STR.

10. Let \(\{a_n\} \subset \mathbb{N}\) be “very lacunary”, that is, assume that \(\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} > 3\). Prove that the set \(\{\alpha \in \mathbb{R} \mid \{a_n \alpha \mod 1\} \text{ is not dense}\}\) is uncountable.