1. Let \( \varphi : G \to \hat{G} \) be a group homomorphism, and suppose that its image has \( n \) elements, that is, the order of \( \varphi(G) \) is equal to \( n \). Prove that \( a^n \in \text{Ker} \varphi \) for every \( a \in G \).

\[
|\varphi(G)| = n \Rightarrow (\varphi(a))^n = \bar{e} \quad \forall a \in G
\]

\[
\Rightarrow \varphi(a^n) = \bar{e} \quad \forall a \in G \quad \Leftrightarrow \quad a^n \in \text{Ker} \varphi \quad \forall a \in G
\]

2. Show that there is no homomorphism from \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) onto \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

Let \( \varphi \) be such a homomorphism, then \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \cong (\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) / \text{Ker} \varphi \).

However, the element \((4,0,0)\) must be in the kernel of \( \varphi \) (because all elements of \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \) have orders 4 or less, and \( \varphi(4,0,0) = \varphi(4(1,0,0)) = 4\varphi(1,0,0) = (0,0,0) \)).

Therefore \( (\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) / \text{Ker} \varphi \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), which is not isomorphic to \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

(by counting elements of order 2 or 4)