Problem 1 (3.1 # 5). $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

So, solutions to $Ax = 0$ are of the form $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t$ for some $t \in \mathbb{R}$

i.e. $\ker(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

NOTE: $\ker(A) \neq \{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \}$

In general, the span of a vector is different than just the set including that vector. (The only exception is $\text{span}\{0\} = \{0\}$)

The solutions to problems 7, 10, 14, 16 are fairly similar.

Problem 2 (3.2 # 1). $W = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \}$

If $W$ is a subspace of $\mathbb{R}^3$ then $\vec{0} \in W$, which implies $0 + 0 + 0 = 1$. But this is impossible, so $W$ is not a subspace.

Problem 3 (3.2 #2). $W = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x \leq y \leq z \}$

$W$ cannot be a subspace since if $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $(-1)\vec{v} \notin W$.

Problem 4 (3.2 #3). $W = \text{im} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ Since $W$ is the image of a linear transformation, it is a subspace.

Problem 5 (3.2 #6). (a) Let $V$ and $W$ be subspaces of some vector space. Then $\vec{0} \in V$ and $\vec{0} \in W$ so $\vec{0} \in V \cap W$. In particular, the intersection is not empty. Let $\vec{a}$
and $\vec{b}$ be in $V \cap W$. Then $\vec{a} + \vec{b} \in V$ and $\vec{a} + \vec{b} \in W$, so $\vec{a} + \vec{b} \in V \cap W$. In addition, for any scalar $k \in \mathbb{R}$, $k\vec{a} \in V$ and $k\vec{a} \in W$, so $k\vec{a} \in V \cap W$. Thus, $V \cap W$ is also a subspace.

(b) Let $V = \text{span}\{\vec{e}_1\}$ and $W = \text{span}\{\vec{e}_2\}$ where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors for $\mathbb{R}^2$. We know that $V$ and $W$ are both subspaces, since $\text{span}\{\vec{v}\}$ is always a subspace for any vector $\vec{v} \in \mathbb{R}^2$. However, $\vec{e}_1 + \vec{e}_2 \notin V \cup W$. 