1. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation with $T(\vec{e}_1) = 2\vec{e}_1 + \vec{e}_2 + 3\vec{e}_3 - 4\vec{e}_4$, $T(\vec{e}_2) = \vec{e}_1 - \vec{e}_2 + 2\vec{e}_3 + 6\vec{e}_4$, and $T(\vec{e}_3) = 4\vec{e}_1 - \vec{e}_2 + 7\vec{e}_3 + 8\vec{e}_4$.

(a) (6 points) Write the standard matrix for $T$. Denote this matrix $A$.
(b) (4 points) What is the rank of $A$?
(c) (4 points) Find all solutions to the equation

$$T(\vec{x}) = \begin{bmatrix} -1 \\
-5 \\
0 \\
26 \end{bmatrix}. $$

(d) (4 points) For what values of $c$ does the equation

$$T(\vec{x}) = \begin{bmatrix} -1 \\
-5 \\
0 \\
c \end{bmatrix}. $$

have at least one solution?

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) (8 points) Complete the definition: The image of $T$ is...
(b) (8 points) Prove that the image of $T$ is a subspace of $\mathbb{R}^m$.

3. (a) (6 points) What is the matrix for counterclockwise rotation of $\mathbb{R}^2$ through an angle $\theta$?

(b) (6 points) For any angle $\theta$, let $L_\theta$ denote the line in $\mathbb{R}^2$ obtained by rotating the $x$-axis counterclockwise about the origin through the angle $\theta$. As we saw in class, the matrix for reflection across $L_\theta$ is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\
\sin 2\theta & -\cos 2\theta \end{bmatrix}. $$

(You don’t have to prove this.) Now, for two angles $\alpha$, $\beta$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation consisting of reflection across $L_\alpha$ followed by reflection across $L_\beta$. Find the matrix for $T$. 

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(c) (3 points) \( T \) is a rotation through some angle. What angle? \( \text{Hint:} \) Use the angle addition formulas for sine and cosine. If you can’t remember them, use your answer for (a) to figure them out, as we’ve discussed in class.

(d) (3 points) Draw a sketch illustrating where \( T \) sends the standard basis vectors in the case where \( \alpha = \pi/4 \) and \( \beta = \pi/2 \).

4. (8 points each) Say whether each of the following functions is linear or not. If it is linear, find its matrix. If it is not linear, give an example that shows why not.

(a) 
\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_3 \\ 2x_1 - 4x_2 + x_3 \\ 100x_2 + 7x_3 \end{bmatrix}
\]

(b) 
\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 + x_3 \\ x_1^3 + 3x_2^2 \end{bmatrix}
\]

5. (8 points each) True or false? If the statement is true, justify it. If it is false, provide a counterexample.

(a) (8 points) If \( A \) is an \( n \times n \) matrix, and there is a vector \( \vec{b} \in \mathbb{R}^n \) for which \( A\vec{x} = \vec{b} \) has a unique solution, then the equation \( A\vec{x} = \vec{c} \) has a unique solution for every \( \vec{c} \in \mathbb{R}^n \).

(b) (8 points) The set consisting of all vectors \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \) with \( xyz = 0 \) is a subspace of \( \mathbb{R}^3 \).

6. (a) (8 points) Find the inverse of the matrix 
\[
A = \begin{bmatrix} 1 & 6 & -1 \\ 0 & -2 & 1 \\ 3 & 3 & 5 \end{bmatrix}.
\]

(b) (8 points) Without doing any additional Gauss-Jordan elimination, find all solutions to the system of equations
\[
\begin{align*}
x + 6y - z &= 3 \\
-2y + z &= 1 \\
3x + 3y + 5z &= -2.
\end{align*}
\]