MATH 15a: Linear Algebra  
Practice Exam 2

Write all answers in your exam booklet. Remember that you must show all work and justify your answers for credit. No calculators are allowed. Good luck!

1. Compute the determinant of each of the following matrices. Indicate clearly the method being used.

(a)  
\[
A = \begin{bmatrix}
0 & 2 & 6 \\
3 & 4 & 5 \\
2 & -2 & 1
\end{bmatrix}
\]

(b)  
\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix}
\]

(Hint: Is this matrix invertible? What does that tell you about its determinant?)

2. Let \(A\) be the matrix  
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & -2 \\
-6 & 1 & -3 & 0 & 9 & -4 \\
2 & 0 & 1 & 0 & -3 & 1 \\
0 & 0 & 0 & 2 & -6 & 6
\end{bmatrix}
\]

(a) If \(A\) is the matrix for a linear transformation \(T: \mathbb{R}^n \to \mathbb{R}^m\), what are \(m\) and \(n\)?

(b) Find a basis for \(\ker A\).

(c) Compute the cosine of the angle between the two vectors you obtained in part (a). Is this angle acute, right, or obtuse?

(d) Apply the Gram-Schmidt algorithm to the basis you obtained in part (a) to find an orthonormal basis for \(\ker A\).

3. Let \(\vec{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}\) and \(\vec{v}_2 = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}\). Let \(V\) be the subspace of \(\mathbb{R}^3\) spanned by \(\vec{v}_1\) and \(\vec{v}_2\).

(a) Find a vector \(\vec{v}_3 \in \mathbb{R}^3\) such that \(\mathcal{B} = \vec{v}_1, \vec{v}_2, \vec{v}_3\) is an orthonormal basis for \(\mathbb{R}^3\).

(b) For the basis \(\mathcal{B}\) found in (a), what is \(S_{\mathcal{B}}^{-1}\)?

(c) For any vector \(\vec{w} \in \mathbb{R}^3\), explain what \([\vec{w}]_\mathcal{B}\) means, and write an equation relating \([\vec{w}]_\mathcal{B}\) and \(\vec{w}\).

(d) For the vector \(\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}\), find \([\vec{w}]_\mathcal{B}\).
4. Determine whether each statement is true or false. If it is true, justify it. If it is false, provide a counterexample.

(a) If \( A \) is a square matrix and \( B = \text{rref}(A) \), then \( \det A = \det B \).

(b) If \( A \) is an \( n \times n \) matrix whose columns are an orthonormal basis of \( \mathbb{R}^n \), then for any \( \vec{x}, \vec{y} \in \mathbb{R}^n \), we have \( \vec{x} \cdot \vec{y} = (A\vec{x}) \cdot (A\vec{y}) \).

5. (a) If \( A \) is an invertible \( n \times n \) matrix, what is \( \det(-A) \) in terms of \( \det(A) \)? Justify.

(b) If \( A \) is a skew-symmetric \( n \times n \) matrix, where \( n \) is odd, show \( A \) is not invertible. (Hint: Use your answer to (a).)
MATH 15a: Linear Algebra
Practice Exam 2, Solutions

1. Compute the determinant of each of the following matrices. Indicate clearly the method being used.

(a) 
\[
A = \begin{bmatrix} 0 & 2 & 6 \\ 3 & 4 & 5 \\ 2 & -2 & 1 \end{bmatrix}
\]

Answer: If you use Sarrus’s rule, you get:
\[
\det A = 2 \cdot 2 \cdot 5 + 6 \cdot 3 \cdot (-2) - 2 \cdot 3 \cdot 1 - 6 \cdot 4 \cdot 2
\]
\[
= 20 + (-36) - 6 - 48
\]
\[
= -70
\]

(b) 
\[
A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}
\]

(Hint: Is this matrix invertible? What does that tell you about its determinant?)

Answer: Compute the RREF of this matrix:
\[
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Since there are two rows of zeroes, this means that the matrix is not invertible, so its determinant must be 0.

2. Let \( A \) be the matrix
\[
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -2 \\ -6 & 1 & -3 & 0 & 9 & -4 \\ 2 & 0 & 1 & 0 & -3 & 1 \\ 0 & 0 & 0 & 2 & -6 & 6 \end{bmatrix}
\]

(a) If \( A \) is the matrix for a linear transformation \( T: \mathbb{R}^n \to \mathbb{R}^m \), what are \( m \) and \( n \)?

Answer: Multiplying by \( A \) takes a 6-vector and spits out a 4-vector, so \( n = 6 \) and \( m = 4 \).
(b) Find a basis for \( \text{ker } A \).

**Answer:** If you put this matrix into RREF, you get

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & -2 \\
-6 & 1 & -3 & 0 & 9 & -4 \\
2 & 0 & 1 & 0 & -3 & 1 \\
0 & 0 & 0 & 2 & -6 & 6
\end{bmatrix}
\implies
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & -2 \\
0 & 1 & -3 & 0 & 15 & -16 \\
0 & 0 & 1 & 0 & -5 & 5 \\
0 & 0 & 0 & 1 & -3 & 3
\end{bmatrix}
\implies
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & -2 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -5 & 5 \\
0 & 0 & 0 & 1 & -3 & 3
\end{bmatrix}
\]

The general solution to the system of equations is:

\[
\begin{align*}
x_1 &= -x_5 + 2x_6 \\
x_2 &= x_6 \\
x_3 &= 5x_5 - 5x_6 \\
x_4 &= 3x_5 - 3x_6 \\
x_5 &= \text{free} \\
x_6 &= \text{free}
\end{align*}
\]

So a basis for \( \text{ker } A \) is

\[
\vec{v}_1 = \begin{bmatrix}
-1 \\
0 \\
5 \\
3 \\
1 \\
0
\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}
2 \\
1 \\
-5 \\
-3 \\
0 \\
1
\end{bmatrix}.
\]

(c) Compute the cosine of the angle between the two vectors you obtained in part (a). Is this angle acute, right, or obtuse?

**Answer:** Using the formula, we have:

\[
\cos \theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|} = \frac{-36}{\sqrt{36} \sqrt{40}} = -3/\sqrt{10}
\]

Since this is a negative number, we see that the angle is obtuse.

(d) Apply the Gram-Schmidt algorithm to the basis you obtained in part (a) to find an orthonormal basis for \( \text{ker } A \).

**Answer:** Since \( \|\vec{v}_1\| = \sqrt{1 + 25 + 9 + 1} = 6 \), so

\[
\tilde{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix}
-1/6 \\
0 \\
5/6 \\
3/6 \\
1/6 \\
0
\end{bmatrix}.
\]
Next, note that \( \vec{v}_2 \cdot \vec{u}_1 = -\frac{2}{6} - \frac{25}{6} - \frac{9}{6} = -6 \), so

\[
\vec{v}_2^\perp = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}
\]

It’s easy to double-check that these are both in \( \ker(A) \), have length 1, and are orthogonal to each other.

3. Let \( \vec{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \). Let \( V \) be the subspace of \( \mathbb{R}^3 \) spanned by \( \vec{v}_1 \) and \( \vec{v}_2 \).

(a) Find a vector \( \vec{v}_3 \in \mathbb{R}^3 \) such that \( \mathcal{B} = \vec{v}_1, \vec{v}_2, \vec{v}_3 \) is an orthonormal basis for \( \mathbb{R}^3 \).

**Answer:** We need to find a unit vector in \( \text{Span}(v_1, v_2)^\perp \), or in other words in the kernel of \( A^T \), where \( A = [\vec{v}_1 | \vec{v}_2] \). So find the reduced row echelon form of \( A^T \):

\[
\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \implies \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}
\]

Thus, \( \vec{v}_3 \) must be of the form \( \begin{bmatrix} -4t \\ t \\ t \end{bmatrix} \). Therefore, \( 1 = \|\vec{v}_3\|^2 = \sqrt{(-4t)^2 + t^2 + t^2} = \pm t\sqrt{18} \), so \( t = \pm \sqrt{2}/6 \). Thus, you can take \( \vec{v}_3 \) to be either

\[
\begin{bmatrix} -2\sqrt{2}/3 \\ \sqrt{2}/6 \\ \sqrt{2}/6 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2\sqrt{2}/3 \\ -\sqrt{2}/6 \\ -\sqrt{2}/6 \end{bmatrix}.
\]

(b) For the basis \( \mathcal{B} \) found in (a), what is \( S_{\mathcal{B}}^{-1} \)?

**Answer:** Let’s take

\[
\vec{v}_3 = \begin{bmatrix} -2\sqrt{2}/3 \\ \sqrt{2}/6 \\ \sqrt{2}/6 \end{bmatrix},
\]

so that

\[
S_{\mathcal{B}} = \begin{bmatrix} 1/3 & 0 & -2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/2 & \sqrt{2}/6 \\ 2/3 & \sqrt{2}/2 & \sqrt{2}/6 \end{bmatrix}.
\]

Since \( S_{\mathcal{B}} \) is an orthogonal matrix, we have

\[
S_{\mathcal{B}}^{-1} = S_{\mathcal{B}}^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ -2\sqrt{2}/3 & \sqrt{2}/6 & \sqrt{2}/6 \end{bmatrix}.
\]
(c) For any vector $\vec{w} \in \mathbb{R}^3$, explain what $[\vec{w}]_B$ means, and write an equation relating $[\vec{w}]_B$ and $\vec{w}$.

**Answer:** Any vector $\vec{w}$ can be written as $\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$ for some unique scalars $c_1, c_2, c_3$. $[\vec{w}]_B$ is defined as the vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. It is related to the standard coordinates of $\vec{w}$ by $\vec{w} = S_B [\vec{w}]_B$.

(d) For the vector $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, find $[\vec{w}]_B$.

**Answer:** Writing the formula from (c) another way,

$$
[\vec{w}]_B = S_B^{-1} \vec{w} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2\sqrt{2} \\ 0 \end{bmatrix}.
$$

4. Determine whether each statement is true or false. If it is true, justify it. If it is false, provide a counterexample.

(a) If $A$ is a square matrix and $B = \text{rref}(A)$, then $\det A = \det B$.

**Answer:** False. The determinant changes when we swap two rows or multiply a row by a scalar. For instance, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $\det A = 2$, while $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\det(\text{rref}(A)) = 1$.

(b) If $A$ is an $n \times n$ matrix whose columns are an orthonormal basis of $\mathbb{R}^n$, then for any $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have $\vec{x} \cdot \vec{y} = (A\vec{x}) \cdot (A\vec{y})$.

**Answer:** True. If columns of $A$ are an orthonormal basis, than $A^T A = I_n$.

Therefore, we can write:

$$
(A\vec{x}) \cdot (A\vec{y}) = (A\vec{x})^T (A\vec{y}) = \vec{x}^T A^T A\vec{y} = \vec{x}^T \vec{y} = \vec{x} \cdot \vec{y}.
$$

5. (a) If $A$ is an invertible $n \times n$ matrix, what is $\det(-A)$ in terms of $\det(A)$? Justify.

**Answer:** Multiplying a single row of $A$ by $-1$ has the effect of multiplying $\det A$ by $-1$ as well. To get $-A$ from $A$, we have to change the sign of $n$ rows, so we multiply $\det A$ by $(-1)^n$. Thus, $\det(-A) = (-1)^n \det A$.

(b) If $A$ is a skew-symmetric $n \times n$ matrix, where $n$ is odd, show $A$ is not invertible.

**Answer:** Skew-symmetric means that $A^T = -A$. The determinant of a matrix and its transpose are always equal. Therefore,

$$
\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)
$$

using the fact that $n$ is odd. Therefore, $\det(A) = 0$, which implies that $A$ is not invertible.