F-theory and the classification of elliptic Calabi-Yau manifolds

FRG Workshop:
Recent progress in string theory and mirror symmetry

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Based in part on

arXiv: 1205.0952 WT
arXiv: 1404.6300 G. Martini, WT
arXiv: 1406.0514 S. Johnson, WT
papers to appear w/ J. Halverson and w/ Y. Wang
Basic structure of F-theory:

Requires an *elliptically fibered* CY3 or CY4, 
\[ \pi : X_3 \rightarrow B_2 \text{ or } \pi : X_4 \rightarrow B_3, \pi^{-1}(p) \cong T^2 \]

Geometry \(\rightarrow\) 6D, 4D supergravity.

To construct an F-theory model:

1. Choose base \(B\)
2. Choose point in CS moduli space
   (Weierstrass model \(y^2 = x^3 + fx + g\) encodes IIB \(\tau = \chi + ie^{-\phi}\) as fun. on \(B\).

A theme of this talk & much research over last five years:
**Focusing on geometry of \(B\) \(\rightarrow\) simplifications and insight**

Primary goals/results:

A) Classifying and enumerating elliptic Calabi-Yau threefolds and fourfolds
B) Generic features of \(\mathcal{N} = 1\) F-theory vacua in 6D and in 4D
What does “generic” mean?

Two distinct aspects:

I. Generic features of vacua with fixed $B$
   *i.e.*, features present for *arbitrary* CS moduli ("non-Higgsable structure")

II. Features present for "most" $B$’s

6D: Finite number of $B$’s, fairly clear global picture
   → almost all $B$’s have "non-Higgsable clusters,” certain typical $G$’s

4D: Picture emerging, *surprisingly parallel to 6D picture*
   but more complicated technically, *many open questions*

First: I and II in 6D, where story is clear. Next: emerging related 4D story
6D F-theory vacua and elliptic Calabi-Yau threefolds

I. Non-Higgsable clusters in 6D

Kodaira singularities in EF over divisor \( S_i \)
\[ \rightarrow \text{gauge factor } G_i \text{ from 7-branes} \]
determined by \( \text{ord}_{S_i}f, g \)

For certain \( B_2 \), \( G \) nontrivial \( \forall \) CS moduli

Example: [Morrison/Vafa]

F-theory on \( \mathbb{F}_3 \leftrightarrow \) heterotic \( E_8 \times E_8 \) on K3 w/ (15, 9) instantons
\[ \Rightarrow SU(3) \text{ with no matter (non-Higgsable)} \]

- In other cases, non-Higgsable matter
  e.g. F-theory on \( \mathbb{F}_7 \leftrightarrow \) heterotic w/ (19, 5) instantons
  \[ \Rightarrow G \text{ is generically } E_7 \text{ w/ } \frac{1}{2} \text{ 56}. \text{Can’t Higgs: can’t satisfy D-term constraints} \]

- When no smooth heterotic dual, \( \exists \) non-Higgsable product groups w/ matter
  e.g. \( G_2 \times SU(2) \) w/ (7 + 1, \( \frac{1}{2} \text{ 2} \) matter
Geometry of non-Higgsable groups

The base $B_2$ is a complex surface.
Contains homology classes of complex curves $C_i$

For $C \cong \mathbb{P}^1 \cong S^2$, local geometry encoded by normal bundle $\mathcal{O}(m)$
\textit{e.g.}, $N_C \cong \mathcal{O}(2) \cong TC$ : deformation has 2 zeros, $C \cdot C = +2$

If $N_C \cong \mathcal{O}(-n)$, $n > 0$, $C$ is rigid (no deformations)

For $\mathcal{O}(-n)$, $n > 2$, base space is so curved that 7-branes must pile up to preserve Calabi-Yau structure on total space $\Rightarrow$ non-Higgsable gauge group
Classification of 6D “Non-Higgsable Clusters” (NHC’s)  

[Morrison/WT]

Clusters of curves imposing generic nontrivial codimension one singularities:

\[
\begin{align*}
-m & \quad (m = 3, 4, 5, 6, 7, 8, 12) \\
\text{su}(3), \text{so}(8), \hat{f}_4 & \\
\text{e}_6, \text{e}_7, \text{e}_8 \\
g_2 \oplus \text{su}(2) & \\
g_2 \oplus \text{su}(2) & \\
\text{su}(2) \oplus \text{so}(7) \oplus \text{su}(2) & 
\end{align*}
\]

- Identified using Zariski decomposition of \(-nK\)
  
  e.g., -12 curve: \(-K \cdot C = -10 \Rightarrow -K = (5/6)C + X \) over \(\mathbb{Q}\)
  
  \(\Rightarrow -4K = 4C + Y_{\text{eff}}, -6K = 5C + Z_{\text{eff}} \Rightarrow \text{e}_8\)

- Any other combination including -3 or below \(\Rightarrow (4, 6)\) at point/curve

NHC’s useful in classification of compact (SUGRA) F-theory models (next) 
and in classification 6D SCFTs [cf. Morrison talk]
6D II. Classification/enumeration of elliptic Calabi-Yau threefolds

Gross: \( \exists \) finite number of topologically distinct elliptic CY threefolds
(up to birational equivalence)

Mathematical minimal model program to classify surfaces:
1. Given surface \( S \), find -1 curve \( C \) \( (C \cdot C = -1) \), blow down \( \rightarrow S' \)
2. Repeat until done \( \Rightarrow \) minimal surface

Grassi: minimal surfaces for \( B_2 \) bases:
\( \mathbb{P}^2, \mathbb{F}_m (m \leq 12) \), Enriques \( (-K \sim \text{trivial}) \)

Program: start with \( \mathbb{P}^2, \mathbb{F}_m \), blow up to get all bases \( B_2 \), constrained by NHC’s

Given all bases, then consider tuned Weierstrass models \( \rightarrow \) all elliptic CY3’s
Found all toric $B_2$  

Generic EF Hodge #’s  

[Morrison/WT, WT]

- 61,539 toric bases (some not strictly toric: -9, -10, -11 curves)
- Reproduces large subset of Kreuzer-Skarke database of CY3 Hodge #’s
- Boundary of “shield” from generic elliptic fibrations over blowups of $\mathbb{F}_{12}$.
- In principle: start with $\mathbb{F}_m$, $\mathbb{P}^2$, blow up/tune → all EF CY3’s
Computed all bases w/ $\mathbb{C}^*$-structure [Martini/WT]

- Preserve only one $\mathbb{C}^*$ (“semi-toric”)
- $\sim 162,404$ distinct bases (including toric)
- Includes 6 CY’s with new Hodge numbers (including above)
- Includes 13 with nonzero Mordell-Weil rank ($r = 1-8$, $r = 3$ above).
- Lesson: including branching, loops does not significantly increase complexity
- In principle: can get all $B_2$’s [Wang/WT to appear: all $B$’s, $h^{2,1} > 200$]
Systematic classification of EFS CY3s: Pick bases $B_2$, tune $G$, matter

Start at large $h^{2,1}$

\[
h^{1,1}(X) = r + T + 2
\]
\[
h^{2,1}(X) = H_{\text{neutral}} - 1 = 272 + V - 29T - H_{\text{charged}}.
\]

- Codimension 1, 2 sing.'s $\rightarrow G$, matter $\rightarrow r, V, H_{\text{ch}} \rightarrow h^{1,1}, h^{1,2}$
- Blowups, tuning from minimal bases $\mathbb{P}^2, \mathbb{F}_m$ decrease $h^{2,1}$

Largest $h^{2,1}$: $\mathbb{F}_{12} \ (h^{1,1} = 11, h^{2,1} = 491) \quad [\implies h^{2,1} \leq 491 \ \forall \ \text{EFS CY3's}]$

\[
\begin{aligned}
\tilde{S} &= S + 12F \\
\cdot S &= -12, \quad S \cdot F = 1 \\
\tilde{S} \cdot \tilde{S} &= +12, \quad F \cdot F = 0
\end{aligned}
\]

Tune $f, g \Rightarrow$ enhanced $G$?

- If tune on $D = aS + bF$:
  - if $D \cdot S \neq 0$, (4, 6) pt.
  - $\Rightarrow$ only can tune on $\tilde{S} = S + 12F$

$SU(2)$ on $\tilde{S}$?

Anomaly cancellation $\Rightarrow H_{\text{ch}} = 82 \times 2, \quad (h^{1,1}, h^{2,1}) = (12, 318) \ldots$
To find further CY’s with $350 \leq h^{1,1} \leq 491$, must blow up

\[
(h^{1,1}, h^{2,1}) = (12, 462)
\]

Again, tuning $G \rightarrow h^{2,1} < 300$ (no $G$ on -1’s); blow up again

\[
C \cdot C = -1 = \frac{1}{6} (4 - x_2)
\]
\[
K \cdot C = -1 = -\frac{1}{3} (8 - x_2/2)
\]
\[
H_{ch} = 10 \times 2
\]

(same CY: $(h^{1,1}, h^{2,1}) = (13, 433)) \rightarrow (h^{1,1}, h^{2,1}) = (14, 416 = 433 - 20 + 3)

Continue in this way . . .
EF CY3’s with $h^{2,1} \geq 350$, $\mathbb{F}_m +$ tuning $\rightarrow$ complete classification [Johnson/WT]

- Matches KS; non-toric + toric at (19, 355); new non-toric below 350
- Empirical data on Calabi-Yau’s suggests: “most” (known) CY’s are elliptic, particularly at large Hodge numbers
Can we complete classification at all $h^{2,1}$?

Limitations on existing technology

- **Constructing all non-toric bases:**
  General algorithm w/Wang, issues remain at large $h^{1,1}$, small $h^{2,1}$.

- **$U(1)$’s [Mordell-Weil]:**
  From global information, hard to compute. Much recent work. [Aluffi/Esole, Morrison/Park, Cvetic/Klevers/Piragua, Mayrhofer/Palti/Weigand . . . ]
  Morrison/Taylor, Cvetic/Klevers/Piragua/WT: $U(1)$’s from Higgsing

- **Tuning high rank $I_{N-1}$ singularities:**
  Algebraically difficult at largest values of $N$ but seems tractable

- **Classifying codimension two singularities:**
  No systematic theory (most challenging math problem?)
Codimension 2 singularities: tricky, not completely classified

Simple case: rank one enhancement [Katz-Vafa]

\[ e.g. \quad \Delta \quad S_i \quad \Delta \quad A_n \quad \rightarrow \quad S(U(N)) \text{ enhanced } \Rightarrow \quad S(U(N + 1)) \]
\[ SU(N + 1) \text{ adjoint } \rightarrow \quad SU(N) \quad \square \]

Similar for \( e.g. A_{N-1} \rightarrow D_N : \quad SU(N) \quad \square \)

Realized by embedding of Dynkin diagrams in singularity

Other cases more complex [Sadov, Morrison/WT, Esole/Yau]

\[ A_{N-1} \Rightarrow A_{2N-1} \]
\[ \text{gives } SU(N) \quad \square + \quad \square \text{ or adj } + 1 \]

Analyze through explicit resolution in specific cases [MT, Esole/Yau]

[Grassi/Halverson/Shaneson: alternative: local deformation?]
“Topology” of matter representations

Some reps are not understood in F-theory but seem okay from supergravity

Example: $SU(N)$

Could be associated with $A_3 \rightarrow \hat{D}_6$

But explicit singularity not known.

Genus classification from anomaly structure [Kumar/Park/WT]

$$2g - 2 = (K + C) \cdot C = \sum_R x_R g_R - 2$$

$$g_R = \frac{1}{12} (2C_R + B_R - A_R)$$

e.g. for “box”, $g_{\text{box}} = (N - 1)(N - 2)/2$; $= 3$ for $N = 4$.

(group theory coefficients: $\text{tr}_R F^2 = A_R \text{tr} F^2$, $\text{tr}_R F^4 = B_R \text{tr} F^4 + C_R (\text{tr} F^2)^2$)

Suggests arithmetic genus 3 singularity in curve $C$ on $B_2$ Confirm?
Structure suggests $\sim$ Kodaira: Rep. theory $\leftrightarrow$ cod. 2 sing.’s [$\sim$HLMS “boxes”?]
“Most” bases $B_2$ have NHC’s

The only bases $B_2$ that lack NHC’s are weak Fano = (generalized) del Pezzo

Ten corresponding Hodge pairs $(2 + T, 272 - 29T)$, $T = 0, \ldots, 9$

16 toric bases out of 61,539 lack NHC’s, 27 semi-toric from 162,404

toric bases, those lacking NHC’s in orange
Upshot for a 6D “phenomenologist”

Physics of a “typical” 6D SUSY F-theory compactification

\[ T \sim 25, \text{ rank } G \sim 35, \]

\[ G \sim (G_2 \times SU(2))^3 \times E_8^2 \times F_4 \times SU(3) \times SO(8) \]

Non-Higgsable gauge groups are “generic” in two senses:

– \( G + \text{ matter are non-Higgsable } \Rightarrow \text{ persist throughout CS moduli space} \)

– Almost all \( B \)'s have non-Higgsable \( G \), matter.
  
  No tuning necessary

• Geometric moduli space = physical moduli space, all branches connected

Now how about 4D? Remarkably, story is closely parallel at level of geometry
4D F-theory compactifications

Story parallel in many ways:

– Compactify on Calabi-Yau fourfold, base $B_3 = \text{complex threefold}$
– Underlying moduli space of EF CY fourfolds on rational $B_3 \sim \{\text{EFCY3's}\}$

But:

– G-flux $\Rightarrow$ superpotential, lifts flat directions
– Brane worldvolume DOF nontrivial $\Rightarrow$ can e.g. break gauge group

Today: focus on underlying fourfold geometry, geometric non-Higgsable structure
4D non-Higgsable clusters [Morrison/WT]
(see also: Anderson/WT, Grassi/Halverson/Shaneson/WT)

At level of geometry, similar to 6D but more complicated

Toric bases: straightforward to compute using dual monomials to rays $v_i$

$$f, g \in \text{span} \{ m : \langle m, v_i \rangle \geq -4, -6 \}$$

Generally: 7-branes wrap divisors = surfaces $S$. Orders of vanishing of $f, g$
depend on normal bundle $N_S$.

Expanding in coordinate $z$, $S = \{ z = 0 \}$,

$$f = f_0 + f_1 z + f_2 z^2 + \cdots$$

Up to leading non-vanishing term,

$$f_k \in \mathcal{O}(-4K_S + (4 - k)N_S)$$
$$g_k \in \mathcal{O}(-6K_S + (6 - k)N_S)$$

Can do computations using geometry of surfaces
Example: non-Higgsable $SU(2)$

Take $S = \mathbb{P}^2$, $-K_S = 3H$.

Choose normal bundle $N = -4H$ (analogue of “-4 curve”)

\[ f_0 \in \mathcal{O}(-4K_S + 4N_S) = \mathcal{O}(-4H), f_1 \in \mathcal{O}(0) \]

\[ g_0 \in \mathcal{O}(-6H), g_1 \in \mathcal{O}(-2H), g_2 \in \mathcal{O}(2H) \]

$\mathcal{O}(nH)$ only has non-vanishing elements when $n \geq 0$ ($nH$ effective)

$\Rightarrow f_0 = g_0 = g_1 = 0 \Rightarrow (1, 2)$ vanishing $\Rightarrow SU(2)$

For global model, consider $\mathbb{P}^1$ bundle over $S = \mathbb{P}^2$, “twist” $= -4H$

Threefold $\tilde{F}_4$, 3D analogue of Hirzebruch surfaces

**Single group clusters:** $SU(2), SU(3), G_2, SO(7), SO(8), F_4, E_6, E_7, E_8$

In particular, cannot have: non-Higgsable $SU(5), SO(10)$

Note: $SO(7)$ nontrivial, $f_2 \in \mathcal{O}(2X) \neq 0, g_3 \in \mathcal{O}(3X) = 0, \ X = -2K + N.$
2-factor non-Higgsable group products

Either as an isolated 2-group cluster, or within larger clusters, the only 2-factor products that can appear are:

\[
G_2 \times SU(2), \quad SO(7) \times SU(2), \quad SU(2) \times SU(2), \\
SU(3) \times SU(2), \quad SU(3) \times SU(3)
\]

Example of constraints: why no \(G_2 \times SU(3)?\) \(((2, 3) + (2, 2) = (4, 5))\)

Follows from monodromy conditions

\(SU(3): g_2\) must be a perfect square. Non-Higgsable \(\rightarrow g_2 = cu^2\), unique \(u\)

\(G_2: \) must have \(f_2 = cv^2, \quad g_3 = dv^3\), unique \(v\)

Assume \(SU(3)\) on divisor \(A = \{z = 0\}\), \(G_2\) on divisor \(B = \{w = 0\}\)

Must have leading term in \(g = z^2w^3\), on \(A\), \(g_2 = w^3 \neq u^2\).
4D: clusters can have complicated structure

Basic issue:
On curve, all points homologous
On surface, many distinct curves
Can support independent matter

Branching

Unlike in 6D, cluster structure can have branching in “quiver diagram”

\[ \begin{array}{c}
\text{g}_2 \\
\text{su}_2 \\
\text{g}_2
\end{array} \]

\[ \begin{array}{c}
C_3 \\
S: c_i \cup c_j
\end{array} \]

\[ p_1 \sim p_2 \sim p_3 \]

e.g. (higher branchings also possible)
Clusters can also have long chains

Similar for $\mathbb{F}_8 : SU(2) \times SU(3)^{11} \times SU(2)$
And closed loops

Multiple blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ⇒

Upshot: local structure restricted (9 groups, 5 products), but global structure can be very complex
Classification of elliptic Calabi-Yau fourfolds

Mathematical minimal models $\rightarrow$ Mori theory. No proofs, some exploration:
- w/Halverson: $\mathbb{P}^1$ bundles over toric bases $B_2$
- w/Wang: Monte Carlo exploration by blowing up/down toric bases
- w/Huang, Wang: bases with large $h^{3,1}$

Structure seems closely analogous to EF CY threefolds

“minimal models” $\sim \mathbb{F}_m$

but more complex

Blow up curves, points:
$h^{3,1} \downarrow, h^{1,1} \uparrow$

Generic EF CY 4:
Lots of $G$, matter
at level of geometry

[Lynker, Schimmrigk, Wisskirchen]
Hypersurfaces in weighted $\mathbb{P}^5$ (1998)
Exploring threefold bases $B_3$: $\mathbb{P}^1$ bundles
[Halverson/WT, to appear]

Consider $B_3 = \mathbb{P}^1$ bundle over a surface $S$.

Characterized by “twist” $T = c_1(\mathcal{L})$, $B_3$ projectivization of $\mathcal{L}$
$T$ = normal bundle of section $\cong S$.

Smooth heterotic dual when $S = $ generalized del Pezzo.

Give non-Higgsable $G_1 \times G_2$ on sections, $G_i \subset E_8$ (no common matter);
Classification of dual smooth heterotic/F-theory models: Anderson/WT

More generally, take $S = B_2$, which supports an elliptic CY3

- Enumerated 177,703 distinct $B_3$ for toric $B_2$
- Each represents distinct divisor $S + N$ geometry
  → Exploration of generic fourfolds and of local geometry
Hodge numbers of $\mathbb{P}^1$ bundle $B_3$’s

- Most of these bases have NHC’s, those without have small Hodge #’s
- Set has atypically small $h^{1,1}$, generically expect more NHC’s
- Contains many “minimal” threefold bases $B_3$
Exploring threefold bases $B_3$: Monte Carlo [Wang/WT, to appear]

Start with e.g. $B_3 = \mathbb{P}^3$, blow up and down randomly
(Random walk on graph: samples vertices proportional to incident edges)

Note: does not connect to all bases, need singular intermediaries
but expect as in 6D, may give decent sampling of bulk
Average rank of NHC $\sim 50$
Some statistics

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{P}^1$ bundles</th>
<th>$\mathbb{P}^1$ bdl. $S_{\pm}$</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(2)$</td>
<td>0.39</td>
<td>0.33</td>
<td>0.47</td>
</tr>
<tr>
<td>$G_2$</td>
<td>0.27</td>
<td>0.18</td>
<td>0.31</td>
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<tr>
<td>$F_4$</td>
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<td>0.17</td>
<td>0.09</td>
</tr>
<tr>
<td>$SU(3)$</td>
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<td>0.04</td>
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<tr>
<td>$SO(8)$</td>
<td>0.003</td>
<td>0.04</td>
<td>0.01</td>
</tr>
<tr>
<td>$E_8$</td>
<td>0.07</td>
<td>0.11</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Frequency of single non-Higgsable factors

Monte Carlo frequency of two-group product factors:

- $G_2 \times SU(2) : 54\%$
- $SU(2) \times SU(2) : 30\%$
- $SU(3) \times SU(2) : 8\%$
- $SU(3) \times SU(3) : 7\%$

Statistics are very rough. Systematic issues with each, only covering subsets
Issues with non-Higgsable clusters for fourfolds

– At level of geometry,

6D: geometric moduli space $\leftrightarrow$ physical moduli space

4D: G-flux (Curvature of $C_3$ in M-theory description)
  $\rightarrow$ superpotential $W(\phi)$ lifts moduli
  $\Rightarrow$ may drive vacua to enhanced symmetry points with $G \supset G_{\text{NHC}}$

– Additional DOF in F-theory not understood

Branes on compact surface: DOF on world-volume

• Fluxes in w-volume can break non-Higgsable $G$ [e.g. in Anderson/WT]

• Non-commuting DOF $[X, X]$ for adjoint scalars $\rightarrow$ higher-dimensional branes

• Branes in curved compact spaces not well described

No clear picture yet for how to merge these DOF with F-theory geometry
One approach: T-branes

[Cecotti/Cordova/Heckman/Vafa, Anderson/Heckman/Katz, Collinucci/Savelli, ...]
Realizing the standard model in F-theory

Ignoring the outstanding issues of G-flux and seven-brane DOF, what are the options for realizing \( G = SU(3) \times SU(2) \times U(1) \)?

1. Tune the whole thing — but not on divisors with NHC’s
2. Tune part of \( G \), get part from NHC; e.g. NH \( SU(3) \), tune \( SU(2) \times U(1) \)
3. Get all of \( G \) from non-Higgsable structure

\[ \text{NHC} \Rightarrow SU(3) \times SU(2) \text{ reasonably common, } \sim 8\% \text{ of } G_1 \times G_2 \text{ in MC study} \]

\[ \text{NHC} \Rightarrow U(1) \text{ open question (some NH } U(1)'s \text{ in 6D [Morrison/Park/WT])} \]

Matter spectrum of NH realization of \( SU(3) \times SU(2) \) matches well to SM, given \( U(1) \), anomaly cancellation

Unification

- \( SU(5), SO(10) \) cannot appear as NHC’s. Can’t enhance NHC \( \rightarrow SU(5) \)
- \( E_6, \ldots \) possible for NHC’s, could break e.g. from fluxes on branes.
Dark matter candidates

Two possibilities:

I) “hidden sector” dark matter, e.g. from a disconnected cluster

II) WIMP dark matter (from $SU(2) \times G, G = SU(2), SU(3), SO(7), SO(8)$)
Conclusions: Systematic control of elliptic Calabi-Yau threefolds

– Can systematically construct elliptic threefolds starting from $\mathbb{F}_m, \mathbb{P}^2$, blowing up and tuning by fixing Weierstrass coefficients, $h^{3,1} \downarrow, h^{1,1} \uparrow$

– Constructions of toric, semi-toric, non-toric bases match KS data well

– Clear bounds on set of threefolds, systematic enumeration from large $h^{2,1}$

– Some technical issues: Mordell-Weil, codimension 2/matter

– Story for fourfolds appears similar, but additional complexity.

Generic appearance of gauge groups and matter leads to new physics questions

Can we realize natural phenomenological scenarios with generic gauge/matter?

Natural mechanisms to produce $U(1)$?

Signals of non-Higgsable dark matter?

How does G-flux and the superpotential interact with non-Higgsable clusters?