1. $f(x) = e^{-2x}$. Note that $f(0) = 1$. Compute and evaluate the appropriate derivatives:

- $f'(x) = -2e^{-2x} \Rightarrow f'(0) = -2$
- $f''(x) = 4e^{-2x} \Rightarrow f''(0) = 4$
- $f'''(x) = -8e^{-2x} \Rightarrow f'''(0) = -8$
- $f^{(4)}(x) = 16e^{-2x} \Rightarrow f^{(4)}(0) = 16$
- $f^{(5)}(x) = -32e^{-2x} \Rightarrow f^{(5)}(0) = -32$

Then $P_5(x) = 1 - 2(x - 0) + \frac{4}{2!}(x - 0)^2 - \frac{8}{3!}(x - 0)^3 + \frac{16}{4!}(x - 0)^4 - \frac{32}{5!}(x - 0)^5$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5.$$

2. $f(x) = \ln(x + 1)$. Note that $f(0) = \ln 1 = 0$. Compute and evaluate the appropriate derivatives:

- $f'(x) = \frac{1}{x + 1} = (x + 1)^{-1} \Rightarrow f'(0) = 1$
- $f''(x) = -(x + 1)^{-2} \Rightarrow f''(0) = -1$
- $f'''(x) = 2(x + 1)^{-3} \Rightarrow f'''(0) = 2$
- $f^{(4)}(x) = -6(x + 1)^{-4} \Rightarrow f^{(4)}(0) = -6$
- $f^{(5)}(x) = 24(x + 1)^{-5} \Rightarrow f^{(5)}(0) = 24$
- $f^{(6)}(x) = -120(x + 1)^{-6} \Rightarrow f^{(6)}(0) = -120$

Then $P_6(x) = 0 + x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 + \frac{24}{5!}x^5 - \frac{120}{6!}x^6$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}.$$

If we write this Taylor polynomial in sigma notation, we get $\sum_{i=1}^{6} \frac{(-1)^{i-1}x^i}{i}$.

3. $f(x) = \cos x$. Note that $f(0) = 1$. Compute and evaluate the appropriate derivatives:

- $f'(x) = -\sin x \Rightarrow f'(0) = 0$
- $f''(x) = -\cos x \Rightarrow f''(0) = -1$
- $f'''(x) = \sin x \Rightarrow f'''(0) = 0$
- $f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(0) = 1$
- $f^{(5)}(x) = -\sin x \Rightarrow f^{(5)}(0) = 0$
- $f^{(6)}(x) = -\cos x \Rightarrow f^{(6)}(0) = -1$
- $f^{(7)}(x) = \sin x \Rightarrow f^{(7)}(0) = 0$
- $f^{(8)}(x) = \cos x \Rightarrow f^{(8)}(0) = 1$

Then $P_8(x) = 1 + 0x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1}{8!}x^8$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.$$
4. We know that $P_5(x) = 1 + 2x - 5x^2 + 2x^3 - 8x^4 + x^5$ is the fifth Taylor polynomial for the function $f(x)$ about $x = 0$, so we know that

$$1 + 2x - 5x^2 + 2x^3 - 8x^4 + x^5 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5.$$ 

Therefore we know the following:

- $f(0) = 1$
- $f'(0) = 2$
- $f''(0) = -5$
- $f'''(0) = 2$
- $f^{(4)}(0) = -8$
- $f^{(5)}(0) = 1$

Now we can answer the questions:

(a) $f(0) = 1$

(b) $f(x)$ is increasing at $x = 0$, since $f'(0) = 2$, which is positive.

(c) Note that $\frac{f''(0)}{2!} = -5$, so $f''(0) = -10$. So $f(x)$ concave down at $x = 0$, since $f''(0)$ is negative.

(d) Note that $\frac{f^{(4)}(0)}{4!} = -8$, so $f^{(4)}(0) = (-8)(4!) = (-8)(24) = -192$.

(e) The equation of the tangent line to the graph of $f(x)$ at $x = 0$ is just first Taylor polynomial for $f(x)$ centered at $x = 0$. Here we can easily see that $P_1(x) = 1 + 2x$.

5. We are looking for the 3rd degree Taylor polynomial of a function $y$ that satisfies the differential equation $\frac{dy}{dx} = y^2(x + 1)$, given that $y(0) = 2$. Because the initial condition is given at $x = 0$, we’ll center the Taylor polynomial at $x = 0$.

Find the first three derivatives of $y(x)$, getting the following:

- $y' = y^2(x + 1)$ (given)
- $y'' = y^2 + 2y(x + 1)$
- $y''' = 2y + 2(x + 1) + 2y = 4y + 2(x + 1)$

Now evaluate at $x = 0$, using the initial condition:

- $y(0) = 2$ (given)
- $y'(0) = (2)^2(0 + 1) = 4$
- $y''(0) = (2)^2 + 2(2)(0 + 1) = 8$
- $y'''(0) = 4(2) + 2(0 + 1) = 10$

So our Taylor polynomial is

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 2 + 4x + \frac{8}{2!}x^2 + \frac{10}{3!}x^3 = 2 + 4x + 4x^2 + \frac{5}{3}x^3.$$
6. Let \( f(x) = \int_1^x e^{\sqrt{t}} \, dt \).

(a) We’re asked to find the second Taylor polynomial for \( f(x) \) about \( x = 1 \). Note that \( f(1) = 0 \). The Second Fundamental Theorem of Calculus tells us that \( f'(x) = e^{\sqrt{x}} \), so \( f'(1) = e \). Then \( f''(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \), so \( f''(1) = \frac{e}{2} \). Therefore

\[
P_2(x) = 0 + e(x - 1) + \frac{e}{2 \cdot 2!}(x - 1)^2 = e(x - 1) + \frac{e}{4}(x - 1)^2.
\]

(b) We’re asked to estimate \( f(4) \). Note that \( P_2(4) \approx f(4) \). Since

\[
P_2(4) = e(4 - 1) + \frac{e}{4}(4 - 1)^2 = 3e + \frac{9e}{4} = \frac{21e}{4},
\]

we conclude that \( \int_1^4 e^{\sqrt{t}} \, dt \approx \frac{21e}{4} \).