Course Overview

1. Key Objectives
2. The Forecast Process
3. US GDP: Unit Root
4. Multi-Period Forecasts
Key Objectives

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2. The Forecast Process
3. US GDP: Unit Root
4. Multi-Period Forecasts
Key Objectives

- The Forecast Process
- US GDP: Unit Root
- Multi-Step Forecasts
- US GDP: Time Trend
- US Unemployment
- Gasoline Supply
- ARMAX
- Rolling & Recursive Forecasts
1. Key Objectives
2. The Forecast Process
3. US GDP: Unit Root
4. Multi-Period Forecasts
The Forecast Process

- Model Identification
- Residual Diagnostics
- Single & Multi-Step Forecasts
- Forecast Confidence Intervals
Examples

- US GDP
- US Unemployment
- US Gasoline Supply
1 Key Objectives

2 The Forecast Process

3 US GDP: Unit Root

4 Multi-Period Forecasts
What Have We Done So Far?

- Looked at GDP visually with tsline; Log-linear
- Looked at ACF and PACF of lgdp
- Estimated AR(1) and AR(1/2); roots looked unstable
- Unit Root Test: Dickey Fuller (which flavor?); Concluded lgdp has unit root
- Now what?
To integrate log real GDP, take the first difference

\[ y_t = \log(GDP_t) - \log(GDP_{t-1}) \]  

1. Model \( y_t \) as an ARMA(p,q) or ARIMA(p,1,q)
2. In words: US GDP growth rates follow ARMA
3. We will still check time trend models too
Identifying the Process for Log Real GDP ($y_t$)

- In Stata: `gen dgdp=D.lgdp` and plot
- Check ACF and PACF; Some Correlation
- Model as AR(1), AR(2), MA(1), MA(2), ARMA(1,1); check roots
- Slight preference to AR models; simpler to estimate
- We will formalize some model selection criterion
Fitting AR Models

- AR models are easy to estimate
- Can use OLS
- In Stata use Lag operator
- This brings up a few theoretical issues
*estimate AR(2) with OLS
reg dgdpl L.dgdpl L2.dgdpl

*now with arima command
arima dgdpl, ar(1/2)

*MA(2) with arima
arima dgdpl, ma(2)

*can also estimate fancy pattern
arima dgdpl, ar(1 3 5)

*can estimate the full model from lgdp
arima lgdp, arima(p,1,q)
Lagged dependent variables:

\[ y_t = \beta_0 + \alpha y_{t-1} + \beta_1 x_t + \eta_t \]

\( \eta_t \) is iid (independently and identically distributed)

\( \eta_t \) is weakly exogenous, \( E[\eta_s|x_t, y_{t-1}] = 0, \quad s \geq t \)

**OLS Properties**

- Parameter estimates biased, consistent
- Asymptotic distributions and standard errors/tests correct
- Many time series models will be like this
These use numerical optimization procedures
Computer searches for **BEST** parameters
Maximum Likelihood Estimation (MLE)
What are Likelihood Functions?

- Estimate the probability of data, given parameters
- Maximize $\text{Prob}(\text{Data} \mid \text{Parameters})$
- Computer over parameters and finds most likely choice
Estimating the Likelihood

Estimated disturbances (ARMA(1,1)):

\[ \hat{e}_t = y_t - \phi y_{t-1} - \theta \hat{e}_{t-1}, \quad \sigma^2 = \text{Var}[\hat{e}_t] \]  

\[ \prod_{t=1}^{T} f_t = \prod_{t=1}^{T} f(\hat{e}_t|\phi, \theta, \sigma) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-\hat{e}_{t}^2}{2\sigma^2}\right] \]  

\[ L = \log(\prod f_t) = -\frac{1}{2} \sum_{t=1}^{T} \log(2\pi\sigma^2) - \sum_{t=1}^{T} \frac{\hat{e}_{t}^2}{2\sigma^2} \]  

\[ L = -\frac{T}{2} \log(2\pi\sigma^2) - \sum_{t=1}^{T} \frac{\hat{e}_{t}^2}{2\sigma^2} \]  

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Identification Methods

- Try Different Models
- Look at Parameter t-Statistics (Standard Errors)
- Look at Residuals (Correlations and Q-Test)
- Look at Information Criterion
Portmanteau Q Test

- Model Residuals (where $x_t$ is vector of ARMA terms)

$$
\epsilon_t = y_t - x_t \beta \quad (7)
$$

- Estimate autocorrelations, $\hat{\rho}_j$ for $L$ lags, for data with sample size $T$. (Note, these are NOT partial autocorrelations)

- Estimate the sum of the squared autocorrelations with fancy weights

$$
Q = T(T-2) \frac{1}{L} \sum_{i=1}^{L} \frac{1}{T-i} \hat{\rho}_i^2 \quad (8)
$$

- For white noise: $Q \rightarrow \chi^2_L$

- Reject white noise (in favor of serial correlation) if $Q > \chi^2_L$
Portmanteau Q Test

* estimate arima(2,1,0) for log gdp
arima lgdp, arima(2,1,0)

* create residuals
predict resid, res

* test for white noise at 20 lags
wntestq resid, lags(20)

* test for white noise at (this case) 40 lags
wntestq resid
Akaike Information Criterion (AIC)

\[ AIC = -2L + 2k \]  \hspace{1cm} (9)

Bayesian Information Criterion (BIC)

\[ BIC = -2L + k \log(T) \]  \hspace{1cm} (10)

- Choose model with smallest AIC or BIC
- Computer will attempt to make L big
- The number of parameters, k, is a penalty
- More parameters result in larger AIC & BIC
Compare 2 models

* model 1
arima lgdp, arima(1,1,0)
predict res1, res
estat aroots
wntestq res1, lags(10)
estat ic

* model 2
arima lgdp, arima(2,1,0)
predict res2, res
estat aroots
wntestq res2, lags(10)
estat ic
One-Step Ahead Forecasts

*one-step ahead fcast for gdp growth rate
arima dgdp, ar(1)
predict yhat

*compare growth rate to fcast
scatter dgdp yhat
tsline dgdp yhat

*one-step ahead fcast for GDP level
arima lgdp, arima(1,1,0)
predict fcast, y

*compare to naive forecast
gen naive = lgdp[_n-1]
tsline lgdp fcast naive if dateq>tq(2003q4)
One-Step Ahead Forecast Comparison

Forecast Examples: Part 1

Patrick Herb (Brandeis University)
95% Confidence Bands: Example AR(1)

\[ E[(y_{t+1} - \hat{y}_{t+1})^2], \quad [\hat{y}_{t+1} - 1.96\hat{\sigma}_\epsilon, \hat{y}_{t+1} + 1.96\hat{\sigma}_\epsilon] \quad (11) \]

\[ y_{t+1} = \rho y_t + \epsilon_t \quad (12) \]
\[ \hat{y}_{t+1} = \hat{\rho} y_t \quad (13) \]
\[ y_{t+1} = \hat{\rho} y_t + \hat{\epsilon}_{t+1} \quad (14) \]

\[ E[(y_{t+1} - \hat{y}_{t+1})^2] = E[(y_{t+1} - \hat{\rho} y_t)^2] \quad (15) \]
\[ = E[\epsilon_{t+1}^2] \quad (16) \]
\[ = \sigma_\epsilon \quad (17) \]
95% Confidence Bands: Example AR(1)

- Two Parts
  - Don’t know $\epsilon_{t+1}$ or $\sigma_\epsilon$
  - Don’t know true $\rho$
  - Forecast uncertainty $\sigma_\hat{y}$
- We will often ignore forecast uncertainty (smaller, more difficult)
Confidence Bands: log(GDP)

ARIMA(1,1,0)

\[ y_t = \log(GDP_t) \] (18)

\[ y_{t+1} = y_t + x_{t+1} \] (19)

\[ x_{t+1} = a + \rho x_t + \epsilon_{t+1} \] (20)

\[
E[(x_{t+1} - E[x_{t+1}|\Omega_t])^2] = E[(x_{t+1} - (a + \rho x_t))^2] \\
= E[\epsilon_t^2] \\
= \sigma^2_{\epsilon} \] (21)

\[ [\hat{y}_{t+1} - 1.96\hat{\sigma}_{\epsilon}, \hat{y}_{t+1} + 1.96\hat{\sigma}_{\epsilon}] \] (24)
Confidence Bands: log(GDP)

```
arima lgdp, arima(1,1,0)
predict yhat, y
predict sigma2, mse
gen sigma = sqrt(sigma2)
gen yl = yhat - 1.96*sigma
gen yh = yhat + 1.96*sigma
tsline lgdp yhat yl yh if dateq>tq(2003q4)
```
Recall the first days of our class?
  - In sample forecasts can look too good
  - Need to check out of sample

Let’s try this with GDP

Easy to do with Stata
*we want to compare mse of in/out sample
use gdp

*model 1: in sample
arima lgdp, arima(1,1,0)
predict fcast1, y
gen in_e2 = (lgdp-fcast1)^2 if dateq>=tq(2000q1)

*model 2: out of sample
arima lgdp if dateq<tq(2000q1), arima(1,1,0)
predict fcast2, y
gen out_e2 = (lgdp-fcast2)^2 if dateq>=tq(2000q1)

*compare mse
sum in_e2 out_e2
Out of Sample Tests

- Need to be able to compare out of sample forecasting performance
- Choosing in and out of sample periods is difficult
- Large in sample estimation
  - Good parameter estimates (in sample)
  - Noisy MSE estimate (out of sample)
- Small in sample estimation
  - Noisy parameter estimates (in sample)
  - Good MSE estimate (out of sample)
Multi-Period Forecasts

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Two Flavors of Multi-Period Forecasts

- Direct: Build forecast for $y_{t+h}$ from $t$
- Recursive: Keep substituting one-step ahead forecasts into your equations
Recursive Multi-Period Forecasts

- Go more than one period into the future
- Replace endogenous variables with forecasts
- Example: AR(1)
- Remember all of our expected value theory from the last few classes?

\[
\begin{align*}
y_{t+1} &= \rho y_t + \epsilon_t \quad (25) \\
\hat{y}_{t+1} &= \hat{\rho} y_t \quad (26) \\
\hat{y}_{t+2} &= \hat{\rho} \hat{y}_{t+1} = \hat{\rho}^2 y_t \quad (27) \\
\hat{y}_{t+3} &= \hat{\rho} \hat{y}_{t+2} = \hat{\rho}^3 y_t \quad (28)
\end{align*}
\]
Dynamic option only works after an appropriate arima estimation
dynamic(tq(2010q1)) starts the true forecast q1 2010
Questions? type help arima postestimation

use gdp
arima lgdp if dateq<=tq(2009q4), arima(1,1,0)
predict yhat , y dynamic(tq(2010q1))
tsline lgdp yhat if dateq>tq(2000q4)
Multi-Step Confidence Bands

\[ y_{t+1} = \rho y_t + \epsilon_{t+1} \]

\[ y_{t+2} = \rho y_{t+1} + \epsilon_{t+2} = \rho^2 y_t + \rho \epsilon_{t+1} + \epsilon_{t+2} \]

\[ E(y_{t+2} - \hat{y}_{t+2})^2 = \rho^2 \sigma^2_\epsilon + \sigma^2_\epsilon \]

\[ E(y_{t+h} - \hat{y}_{t+h})^2 = \sigma^2_\epsilon \sum_{i=0}^{h-1} (\rho^2)^i \]

For big \( h \),

\[ E(y_{t+h} - \hat{y}_{t+h})^2 = \sigma^2_\epsilon \sum_{i=0}^{\infty} (\rho^2)^i = \frac{1}{1 - \rho^2} \sigma^2_\epsilon = \text{Var}[y_t] \]
Finally: A Real Forecast

- What does our model say about future GDP?
- How would you produce actual forecasts? Need **tsappend**

```plaintext
use gdp
*use full sample
arima lgdp, arima(1,1,0)
*add 8 quarters to the data set
tsappend , add(8)

*try without dynamic
predict yhat1, y

*now try dynamic with date
predict yhat, y dynamic(tq(2016q1))
tsline lgdp yhat if dateq>tq(2001q4)
```
Real Forecast of log(GDP)

Forecast Examples: Part 1

Patrick Herb (Brandeis University)