Solutions to Self-Quiz on Differential Equations

1. Note that \( \frac{dy}{dx} = A + 2Bx \), so \( \frac{d^2 y}{dx^2} = 2B \). Plug these, along with \( y = Ax + Bx^2 \), into the differential equation:

\[
x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^2(2B) - 2x(A + 2Bx) + 2(Ax + Bx^2)
\]

\[
= 2Bx^2 - 2Ax - 4Bx^2 + 2Ax + 2Bx^2 = 0.
\]

This checks, so \( y = Ax + Bx^2 \) is indeed a solution.

2. Separate the variables and integrate, getting

\[
\int y^2 \, dy = \int e^x \, dx \Rightarrow \frac{y^3}{3} + C_1 = e^x + C_2 \Rightarrow \frac{y^3}{3} = e^x + C \Rightarrow y^3 = 3e^x + C.
\]

So the general solution is \( y = \sqrt[3]{3e^x + C} \).

3. Separate the variables and integrate, getting

\[
\int \frac{dy}{y^2 + 1} = \int (1 + 3x^2) \, dx \Rightarrow \tan^{-1} y + C_1 = x + x^3 + C_2 \Rightarrow \tan^{-1} y = x + x^3 + C.
\]

It’s easiest to solve for \( C \) at this point. We get \( \tan^{-1}(1) = 0 + C \Rightarrow C = \frac{\pi}{4} \). Therefore \( \tan^{-1} y = x + x^3 + \frac{\pi}{4} \). To finish, solve for \( y \) by taking the tangent of both sides; getting \( \tan(\tan^{-1} y) = \tan(x + x^3 + \frac{\pi}{4}) \Rightarrow y = \tan(x + x^3 + \frac{\pi}{4}) \).

4. Separate the variables and integrate, getting

\[
\int \frac{1}{y} \, dy = \int 2x \, dx \Rightarrow \ln \left| y \right| = x^2 + C \Rightarrow \left| y \right| = e^{x^2 + C} \Rightarrow y = \pm e^{x^2 + C} \Rightarrow y = \pm e^{x^2} e^C.
\]

Note that \( e^C \) is a constant; let’s call it \( A \) for simplicity (we did something similar when we solved the differential equation \( \frac{dy}{dt} = ky \) in class). So now we have

\[
y = Ae^{x^2},
\]

where \( A \) is a constant. We know that when \( x = \sqrt{\ln 5} \), \( y = 100 \). So

\[
100 = Ae^{(\sqrt{\ln 5})^2} \Rightarrow 100 = Ae^{\ln 5} \Rightarrow 100 = 5A \Rightarrow A = 20.
\]

So the specific solution to the differential equation is \( y = 20e^{x^2} \).

5. Let \( y \) be the amount (in milligrams) of the drug present in the bloodstream and let \( t \) be the time (in hours) that has passed since the injection was given. Since the rate at which the drug leaves the bloodstream is proportional to the square root of the amount of the drug present in the bloodstream, we get the following differential equation:

\[
\frac{dy}{dt} = k \sqrt{y} \quad \text{for some constant } k.
\]
This differential equation is separable, so separate the variables and integrate, getting

$$\int \frac{dy}{\sqrt{y}} = \int k \, dt, \quad \text{so} \quad 2\sqrt{y} = kt + C.$$  

It now remains to solve for $C$ and $k$. It is easier, algebraically, to do this now, before we solve for $y$. The initial dose, at time $t = 0$, was 16 milligrams, so when $t = 0$, $y = 16$. So

$$2\sqrt{16} = k \cdot 0 + C \Rightarrow C = 8 \quad \text{so} \quad 2\sqrt{y} = kt + 8.$$  

We also know that when $t = 2$, $y = 9$. Plugging these values into $2\sqrt{y} = kt + 8$ gives $k = -1$. (It makes sense that $k$ is negative, since the amount of the drug in the bloodstream is decreasing).

Therefore $2\sqrt{y} = -t + 8$. Now solve for $y$, getting $y = \left(\frac{-t + 8}{2}\right)^2$. Hence the solution to the differential equation equation is

$$y = \left(\frac{8 - t}{2}\right)^2 \quad \text{or} \quad y = \left(-\frac{1}{2}t + 4\right)^2.$$  

6. (a) Let $y$ be the size of the population. The colony is growing at a rate directly proportional to the fourth root of the population, so we get the following differential equation:

$$\frac{dy}{dt} = k\sqrt[4]{y},$$

where $k$ is a constant. This differential equation is separable, so separate the variables and integrate, getting

$$\int \frac{dy}{\sqrt[4]{y}} = \int k \, dt \Rightarrow \int y^{-\frac{1}{4}} \, dy = \int k \, dt \Rightarrow \frac{4}{3}y^{\frac{3}{4}} = kt + C.$$  

It’s easiest to solve for $C$ and $k$ at this point. Since $y = 1$ when $t = 0$, we get

$$\frac{4}{3} \cdot 1 = k \cdot 0 + C \Rightarrow C = \frac{4}{3}.$$  

Now we have $\frac{4}{3}y^{\frac{3}{4}} = kt + \frac{4}{3}$. Since $y = 16$ when $t = 2$, we get

$$\frac{4}{3}(16)^{\frac{3}{4}} = k \cdot 2 + \frac{4}{3} \Rightarrow \frac{4}{3} \cdot 8 = 2k + \frac{4}{3} \Rightarrow \frac{28}{3} = 2k \Rightarrow k = \frac{14}{3}.$$  

Therefore $\frac{4}{3}y^{\frac{3}{4}} = \frac{14}{3}t + \frac{4}{3}$, so $y^{\frac{3}{4}} = \frac{7}{2}t + 1$. Finally, solve for $y$, getting

$$y = \left(\frac{7}{2}t + 1\right)^{\frac{4}{3}}.$$  

(b) To find out how many cells there are after 18 hours, plug $t = 18$ into the function $y = \left(\frac{7}{2}t + 1\right)^{\frac{4}{3}}$, getting $y = (64)^{\frac{4}{3}} = \left(\sqrt[3]{64}\right)^4 = 4^4 = 256$ cells.